NON-EXISTENCE OF $(76, 30, 8, 14)$ STRONGLY REGULAR GRAPH AND SOME STRUCTURAL TOOLS

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Abstract. Our main result is the non-existence of strongly regular graph with parameters $(76, 30, 8, 14)$. We heavily use Euclidean representation of a strongly regular graph, and develop a number of tools that allow to establish certain structural properties of the graph. In particular, we give a new lower bound for the number of 4-cliques in a strongly regular graph.

1. Introduction

Let $G = (V, E)$ be a finite, undirected, simple graph with vertices V and edges E. The graph G is *strongly regular* with parameters (v, k, λ, μ) if G is k-regular on v vertices, such that any two adjacent vertices have λ common neighbors, and any two non-adjacent vertices have μ common neighbors. It is not known in general for which parameters (v, k, λ, μ) strongly regular graphs exist. One can easily deduce certain necessary conditions on the parameters (see Section 2), but the pattern of the known results is still far from being understood, see [Bro] for a list of results for $v \leq 1300$. There are only three unknown cases for $v < 76$, namely, $(65, 32, 15, 16)$, $(69, 20, 7, 5)$, and $(75, 32, 10, 16)$. The following theorem is our main result which settles the next unknown case.

Theorem 1.1. *There is no strongly regular graph with parameters* $(76, 30, 8, 14)$.

Some numerical evidence for non-existence of this graph was given in [Deg07, Section 6.1.6, p. 204], which involved a significant (and not exhaustive) computer search.

Let us outline the structure of the proof. Assuming the existence of such a graph G, we first show that it must contain a 4-clique (complete graph on 4 vertices) as a subgraph. This is a crucial first step, which then allows to show that G contains a much larger "nice" induced subgraph: either a $(40, 12, 2, 4)$ strongly regular graph, or a 16-coclique (empty graph on 16

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vertices), or a complete bipartite graph $K_{6,10}$ (two parts of 6 and 10 vertices, with an edge between vertices if and only if the vertices are from different parts). In what follows, by a subgraph we always mean the induced subgraph. Each of these three cases is treated differently but ultimately leads to a contradiction. The last two cases were completed using machine-assisted searches with total running time of under two hours on a personal computer. We would like to emphasize that our methods are primarily analytical and establish strong structural properties of the graph. Use of computer is minor as we need to run a very insignificant verification.

To establish such strong structural properties of G, we developed a number of tools which use the Euclidean representation of a strongly regular graph as a system of unit vectors in a finite-dimensional Euclidean space (see Section 2 for the definitions). The tools are presented in Section 4 with complete statements and proofs. Most of the statements involve quite technical computations, so for convenience we implemented these computations as easy-to-use functions in SageMath ([S$^+$13]) computer algebra system, see [BPR] for downloadable worksheets. While our tools may be applied for any strongly regular graph, we observed non-trivial corollaries mostly for graphs which have 2 as an eigenvalue.

Another result of possibly independent interest is a lower bound on the number of 4-cliques in a strongly regular graph, see Theorem 3.3. The proof is based on Euclidean representation and on a relation to spherical harmonic polynomials. The bound that we obtain contains quite lengthy expression involving parameters of the strongly regular graph, so we provide a table of the resulting (numerical) bounds on the number of 4-cliques for all admissible $v \leq 1300$ in [BPR]. While for some situations this estimate may be trivial or easy-to-obtain by other methods, for our $(76, 30, 8, 14)$ strongly regular graph it shows that there exist at least 39 4-cliques, and we do not know any other proof for this special case (for the proof of Theorem 1.1 we only need the existence of one 4-clique).

The paper is organized as follows. We describe some preliminaries and notations in Section 2. Then we establish our lower bound on the number of 4-cliques (Theorem 3.3) in Section 3. Auxiliary tools arising from the Euclidean representation of the strongly regular graphs are stated and proved in Section 4, which also includes some specific computations for the case $(76, 30, 8, 14)$. In Section 5.1, we reduce Theorem 1.1 to one of the three main cases, which are treated in Sections 6, 7, and 8.

2. Preliminaries

Throughout this section let $G = (V, E)$ be a strongly regular graph (SRG) with parameters (v, k, λ, μ). By $N(i) := \{j : (i, j) \in E\}$ we will denote the set of all neighbors of a vertex $i \in V$.

2.1. **Spectral properties.** The incidence matrix A of G has the following properties:

$$AJ = kJ, \quad \text{and} \quad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J, \tag{2.1}$$

where I is the identity matrix and J is the matrix with all entries equal to 1. These conditions imply that

$$(v - k - 1)\mu = k(k - \lambda - 1). \tag{2.2}$$

Moreover, the matrix A has only three eigenvalues: k of multiplicity 1, a positive eigenvalue

$$r = \frac{1}{2}\left(\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right) \tag{2.3}$$

of multiplicity

$$f = \frac{1}{2}\left(v - 1 - \frac{2k + (v - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}\right), \tag{2.4}$$

and a negative eigenvalue

$$s = \frac{1}{2}\left(\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right) \tag{2.5}$$

of multiplicity

$$g = \frac{1}{2}\left(v - 1 + \frac{2k + (v - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}\right). \tag{2.6}$$

Clearly, f and g should be integer. This together with (2.2) gives a family of suitable parameters (v, k, λ, μ) for strongly regular graphs. The reader can refer to [BH12, Section 9.1.5] for the proofs of the above relations.

For $(v, k, \lambda, \mu) = (76, 30, 8, 14)$, we have $r^f = 2^{57}$ and $s^g = (-8)^{18}$.

2.2. **Euclidean representation.** Now we will construct an Euclidean representation of G in $\mathbb{R}^g$. Take the columns $\{y_i : i \in V\}$ of the matrix $A - rI$ and let $x_i := z_i/\|z_i\|$, where

$$z_i = y_i - \frac{1}{|V|}\sum_{j \in V} y_j, \quad i \in V,$$

and $\|z_i\| := (z_i \cdot z_i)^{1/2}$. Here and below $x \cdot y$ will denote the dot product of x and y in the corresponding Euclidean space, and $|V|$ denotes the number of elements in a set V.

It is straightforward to verify that this set of vectors $\{x_i : i \in V\} \subset \mathbb{R}^g$ satisfies the following two properties. First, there are only two possible non-trivial values of the dot product depending on adjacency:

$$x_i \cdot x_j = \begin{cases} 1, & \text{if } i = j, \\ p, & \text{if } i \text{ and } j \text{ are adjacent}, \\ q, & \text{otherwise}, \end{cases} \tag{2.7}$$

where p and q are real numbers from the interval $(-1, 1)$, namely

$$p = s/k, \quad \text{and} \quad q = -(s+1)/(v-k-1). \tag{2.8}$$

The second property is that the set $\{x_i : i \in V\}$ forms a spherical 2-design, i.e.,

$$\sum_{i \in V} x_i = 0, \quad \text{and} \quad \sum_{i \in V}(x_i \cdot y)^2 = \frac{|V|}{g} \quad \text{for any } y,\ \|y\| = 1. \tag{2.9}$$

For more information on the relations between the Euclidean representation of strongly regular graphs and spherical designs see, e.g., [Cam04].

One of the key facts that we will use for developing our tools is the following evident proposition.

Proposition 2.1. *Each subset $\{x_i : i \in U\}$, where $U \subset V$, has a non-negative definite Gram matrix $(x_i \cdot x_j)_{i,j \in U}$ of rank equal to the rank of the linear span of $\{x_i : i \in U\}$, which is at most g. If A is the adjacency matrix of the subgraph induced by U, then $(x_i \cdot x_j)_{i,j \in U} = pA + I + q(J - I - A)$.*

Another observation that we will use is that

$$x_i = \frac{1}{kp}\sum_{j \in N(i)} x_j, \quad \text{for each } i \in V. \tag{2.10}$$

Indeed, for arbitrary $l \in G$, it straightforward to check that $(kpx_i - \sum_{j\in N(i)} x_j)\cdot x_l = 0$ (using, in particular, (2.2) and (2.8)).

Remark 2.2. One can construct an Euclidean representation of G in $\mathbb{R}^f$ which will posses similar properties. This can be done by considering the complement of G, which is a strongly regular graph with parameters $(v, v-1-k, v-2k+\mu-2, v-2k+\lambda)$; then f and g interchange.

For $(v, k, \lambda, \mu) = (76, 30, 8, 14)$, the Euclidean representation in $\mathbb{R}^{18}$ has dot products $(p, q) = (-\frac{4}{15}, \frac{7}{45})$, and the Euclidean representation in $\mathbb{R}^{57}$ (obtained through the complement) has dot products $(p, q) = (\frac{1}{15}, -\frac{1}{15})$, see (2.7) and (2.8).

2.3. **Vertex partitions.** Let $\pi = \{G_1, \dots, G_l\}$ be a partition of a subset $\widetilde{V} \subset V$ of the vertices of a graph $G = (V, E)$. We define the *edge matrix* $\mathcal{E}_\pi = (a_{i,j})_{i,j=1}^l$ of the partition π by assigning $a_{i,j}$ to be the number of edges $(x, y) \in E$ such that $x \in G_i$ and $y \in G_j$. In particular, $a_{i,i}$ is the number of edges in the subgraph induced by G_i. As $\mathcal{E}_\pi$ is symmetric, in what follows we will often not list the entries that are below main diagonal.

A partition π is *equitable* if there exist non-negative integers $b_{i,j}$, $1 \le i, j \le l$, such that any vertex $x \in G_i$ has exactly $b_{i,j}$ neighbors in G_j, regardless of the choice of x. The *degree matrix* of π is $\mathcal{D}_\pi := (b_{i,j})$. Clearly, there is a relation to the entries of the edge matrix: $a_{i,i} = b_{i,i}|G_i|/2$, and $a_{i,j} = |G_i|b_{i,j} = |G_j|b_{j,i}$ for $i \ne j$. Any graph possesses an equitable partition of all vertices where each part consists of exactly one vertex: $l = |V|$ and $|G_i| = 1$, then $\mathcal{D}_\pi$ coincides with the adjacency matrix of G. We are primarily interested in less trivial equitable partitions, and often this will happen on a relatively small subset of vertices $\widetilde{V}$.

For a strongly regular graph G, some non-trivial relations on $\mathcal{E}_\pi$ for any partition π and on $\mathcal{D}_\pi$ for any equitable partition π are derived from the Euclidean representation of G in Section 4.

3. Lower bound on the number of 4-cliques

We begin with some preliminaries from harmonic analysis.

3.1. **Spherical harmonic polynomials.** A homogeneous real algebraic polynomial of degree t on $\mathbb{R}^n$ is a real linear combination of monomials $x_1^{t_1} \dots x_n^{t_n}$, where $t_1, \dots, t_n$ are non-negative integers with sum t. Let Δ be the Laplace operator in $\mathbb{R}^n$

$$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.$$

An algebraic polynomial P on $\mathbb{R}^n$ is said to be harmonic if $\Delta P = 0$. For integer $t \geq 1$, the restriction to the unit sphere S^{n-1} in $\mathbb{R}^n$ of a homogeneous harmonic polynomial of degree t is called a spherical harmonic of degree t. The vector space of all spherical harmonics of degree t will be denoted by $\mathcal{P}_{n,t}$. Various properties of spherical harmonics can be found, for example, in [DX13, Chapter 1].

We can equip $\mathcal{P}_{n,t}$ with the inner product

$$\langle P, Q\rangle = \int_{S^{n-1}} P(x)Q(x)\, d\mu_n(x),$$

where μ_n is the Lebesgue measure on S^{n-1} normalized by $\mu_n(S^{n-1}) = 1$. By the Riesz representation theorem, for each point $x \in S^{n-1}$, there exists a unique polynomial $P_x \in \mathcal{P}_{n,t}$ satisfying

$$\langle P_x, Q\rangle = Q(x) \quad \text{for all} \quad Q \in \mathcal{P}_{n,t}.$$

This spherical harmonic P_x can be conveniently expressed using the Gegenbauer polynomials $C_t^{(\alpha)}(\xi)$ with $\alpha = (n-2)/2$. The polynomials $C_t^{(\alpha)}(\xi)$ are orthogonal on $[-1, 1]$ with the weight $(1-\xi^2)^{\alpha-1/2}$, and can be defined from the generating function

$$\frac{1-z^2}{(1-2\xi z+z^2)^{\alpha+1}} = \sum_{t=0}^{\infty} \frac{t+\alpha}{\alpha} C_t^{(\alpha)}(\xi) z^t,$$

or in many other ways [DX13, Appendix B.2]. Now, for $x, y \in S^{n-1}$, we have (see, e.g., [DX13, Lemma 1.2.5, Theorem 1.2.6]):

$$\langle P_x, P_y\rangle = Z_{n,t}(x\cdot y), \quad \text{where} \quad Z_{n,t}(\xi) = \frac{2t+n-2}{n-2} C_t^{((n-2)/2)}(\xi).$$

Note that $\langle P_x, P_y\rangle$ depends only on $x \cdot y$, which also easily follows from the fact that the space $\mathcal{P}_{n,t}$ is rotation invariant. The spherical harmonic $Z_{n,t}(x \cdot y)$ (with fixed $x \in S^{n-1}$ as a function of $y \in S^{n-1}$) is referred to as a zonal harmonic.

Using the Cauchy-Schwarz inequality in $\mathcal{P}_{n,t}$, for any finite sets of points $\{x_i\}_{i\in\mathcal{I}}$ and $\{y_j\}_{j\in\mathcal{J}}$ from S^{n-1}, we obtain

$$\begin{aligned}\left(\sum_{i\in\mathcal{I}, j\in\mathcal{J}} \langle P_{x_i}, P_{y_j}\rangle\right)^2 &= \left\langle \sum_{i\in\mathcal{I}} P_{x_i}, \sum_{j\in\mathcal{J}} P_{y_j}\right\rangle^2 \\ &\leq \left\langle \sum_{i\in\mathcal{I}} P_{x_i}, \sum_{i\in\mathcal{I}} P_{x_i}\right\rangle \left\langle \sum_{j\in\mathcal{J}} P_{y_j}, \sum_{j\in\mathcal{J}} P_{y_j}\right\rangle \\ &= \sum_{i,i'\in\mathcal{I}} \langle P_{x_i}, P_{x_{i'}}\rangle \sum_{j,j'\in\mathcal{J}} \langle P_{y_j}, P_{y_{j'}}\rangle.\end{aligned}$$

Rewriting this in terms of the polynomials $Z_{n,t}$, we obtain (recall that $x_i, y_j \in S^{n-1}$)

$$\left(\sum_{i\in\mathcal{I},j\in\mathcal{J}} Z_{n,t}(x_i \cdot y_j)\right)^2 \le \left(\sum_{i,i'\in\mathcal{I}} Z_{n,t}(x_i \cdot x_{i'})\right)\left(\sum_{j,j'\in\mathcal{J}} Z_{n,t}(y_j \cdot y_{j'})\right). \tag{3.1}$$

This inequality with $t = 4$ and proper choice of x_i, y_j arising from the Euclidean representation of a strongly regular graph will play a crucial role in the next subsection.

Remark 3.1. The inequality (3.1) is valid whenever the function $Z_{n,t}$ is positive definite in S^{n-1} in terminology of [Sch42]. Any finite positive linear combination of Gegenbauer polynomials $C_t^{((n-2)/2)}$ (with fixed n and different t) is positive definite in S^{n-1}. On the other hand, any positive definite function in S^{n-1} is a series of Gegenbauer polynomials with non-negative coefficients, see [Sch42, Theorem 1].

3.2. **Bound.** Let $G = (V, E)$ be a strongly regular graph with parameters (v, k, λ, μ). Recall that for any vertex $x \in V$ we let $N(x)$ be the set of all neighbors of x. Also let $N'(x)$ be the set of non-neighbors of x, i.e. $N'(x) = V \setminus (\{x\} \cup N(x))$. For any adjacent vertices x and y, we consider the following vertex partition of $V \setminus \{x, y\}$

$$\pi := \{N(x) \cap N(y), N'(x) \cap N(y), N(x) \cap N'(y), N'(x) \cap N'(y)\}.$$

Let $\mathcal{E}_\pi = (a_{i,j})$ be the edge matrix of π. Now we will prove a statement expressing all entries of $\mathcal{E}_\pi$ using the parameters of our strongly regular graph and the value of $a_{1,1}$. While the proof is rather straightforward using strong regularity of G, we include it for completeness.

Proposition 3.2. *With the above notations, let $a := a_{1,1}$. We have*

$$\mathcal{E}_\pi = \begin{pmatrix} a & \lambda(\lambda-1)-2a & \lambda(\lambda-1)-2a & \lambda(k-2\lambda)+2a \\ & \frac{\lambda(k-2\lambda)}{2}+a & (\mu-1)(k-\lambda-1)-\lambda(\lambda-1)+2a & (k-\mu)(k-\lambda-1)-\lambda(k-2\lambda)-2a \\ & & \frac{\lambda(k-2\lambda)}{2}+a & (k-\mu)(k-\lambda-1)-\lambda(k-2\lambda)-2a \\ & & & \frac{k(v-2k+\lambda)}{2}-(k-\mu)(k-\lambda-1)+\frac{\lambda(k-2\lambda)}{2}+a \end{pmatrix}.$$

Proof. By the definition of G we have $|N(x) \cap N(y)| = \lambda$, hence $|N'(x) \cap N(y)| = |N(x) \cap N'(y)| = k - \lambda - 1$, and $|N'(x) \cap N'(y)| = v - 2k + \lambda$. For each $z \in V$ denote by a_z, b_z, and c_z the number of its neighbors in $N(x) \cap N(y)$, in $N'(x) \cap N(y)$, and in $N'(x) \cap N'(y)$ respectively.

First we compute $a_{1,2}$, which is the number of edges between $N(x) \cap N(y)$ and $N'(x) \cap N(y)$. For any vertex $z \in N(x) \cap N(y)$, z and y have exactly λ common neighbors in G, therefore

$$a_z + b_z = \lambda - 1, \quad \text{and hence} \sum_{z\in N(x)\cap N(y)} a_z + \sum_{z\in N(x)\cap N(y)} b_z = \lambda(\lambda - 1).$$

So,

$$a_{1,2} = \sum_{z \in N(x) \cap N(y)} b_z = \lambda(\lambda - 1) - 2a.$$

Due to symmetry of the arguments, the same computation applies to $a_{1,3}$ yielding the same result, so $a_{1,3} = a_{1,2}$.

Next we compute $a_{2,2}$, which is the number of edges in $N'(x) \cap N(y)$. For any vertex $z \in N'(x) \cap N(y)$, z and y have exactly λ common neighbors in V, so

$$a_z + b_z = \lambda, \quad \text{and hence} \sum_{z \in N'(x) \cap N(y)} a_z + \sum_{z \in N'(x) \cap N(y)} b_z = \lambda(k - \lambda - 1).$$

Already computed expression for $a_{1,2}$ means that

$$\sum_{z \in N'(x) \cap N(y)} a_z = \lambda(\lambda - 1) - 2a. \tag{3.2}$$

Hence,

$$a_{2,2} = \frac{1}{2} \sum_{z \in N'(x) \cap N(y)} b_z = \frac{\lambda(k - 2\lambda)}{2} + a.$$

Again, we automatically obtain $a_{3,3} = a_{2,2}$.

Now we consider $a_{2,3}$ (the number of edges between $N'(x) \cap N(y)$ and $N(x) \cap N'(y)$). As before, for any vertex $z \in N'(x) \cap N(y)$, z and y have exactly μ common neighbors in V, so

$$a_z + b_z = \mu - 1, \quad \text{and hence} \sum_{z \in N'(x) \cap N(y)} a_z + \sum_{z \in N'(x) \cap N(y)} b_z = (\mu - 1)(k - \lambda - 1).$$

Using (3.2), we conclude

$$a_{2,3} = \sum_{z \in N'(x) \cap N(y)} b_z = (\mu - 1)(k - \lambda - 1) - \lambda(\lambda - 1) + 2a.$$

Next we count $a_{2,4}$ (the number of edges between $N'(x) \cap N(y)$ and $N'(x) \cap N'(y)$). Counting all edges coming from $N'(x) \cap N(y)$, we get immediately

$$a_{2,4} = (k - 1)(k - \lambda - 1) - a_{2,3} - a_{2,1}$$

$$= (k - 1)(k - \lambda - 1) - (\mu - 1)(k - \lambda - 1) + \lambda(\lambda - 1) - 2a - \lambda(k - 2\lambda) - 2a - \lambda(\lambda - 1) + 2a$$

$$= (k - \mu)(k - \lambda - 1) - \lambda(k - 2\lambda) - 2a.$$

Symmetry gives $a_{3,4} = a_{2,4}$.

To compute $a_{1,4}$ (the number of edges between $N(x) \cap N(y)$ and $N'(x) \cap N'(y)$), we count all edges coming from $N(x) \cap N(y)$. We get

$$a_{1,4} = \lambda(k-2) - a_{1,2} - a_{1,3} = \lambda(k - 2\lambda) + 2a.$$

It remains to evaluate $a_{4,4}$, which is the number of edges in $N'(x) \cap N'(y)$. We count all edges coming from $N'(x) \cap N'(y)$, and obtain

$$\begin{aligned} a_{4,4} &= \frac{1}{2}(k(v-2k+\lambda) - a_{1,4} - a_{2,4} - a_{3,4}) \\ &= \frac{k(v-2k+\lambda)}{2} - (k-\mu)(k-\lambda-1) + \frac{\lambda(k-2\lambda)}{2} + a. \end{aligned}$$

Proposition 3.2 is proved. □

Our intention will be to apply (3.1), where we choose $x_i \in \mathbb{R}^g$ to be the Euclidean representation of $i \in V$ (satisfying (2.8)) for all $|V| = v$ vertices of the graph, and $y_j := \frac{x_{j^{(1)}} + x_{j^{(2)}}}{\|x_{j^{(1)}} + x_{j^{(2)}}\|}$ for all $|E| = \frac{vk}{2}$ edges $j \in E$, here j joins the vertices $j^{(1)}, j^{(2)} \in V$. Note that $\|x_{j^{(1)}} + x_{j^{(2)}}\| = \sqrt{2+2p}$. We proceed by computing and introducing notations for certain components of (3.1). Note that in our settings n of (3.1) is g.

Fixing a vertex $i \in V$, we can have three possibilities: $i' = i$, $i' \in N(i)$, or $i' \in N'(i)$. Then, clearly,

$$\sum_{i,i' \in V} Z_{g,t}(x_i \cdot x_{i'}) = v(Z_{g,t}(1) + kZ_{g,t}(p) + (v-k-1)Z_{g,t}(q)) =: \Psi_A(v,k,\lambda,\mu,t). \tag{3.3}$$

Next, we fix a vertex $i \in V$. There are k edges which join i and a vertex in $N(i)$. There are $\frac{k\lambda}{2}$ edges joining some two vertices of $N(i)$. Next, some $(v-k-1)\mu$ edges are between $N(i)$ and $N'(i)$. Finally, we have $\frac{(v-k-1)(k-\mu)}{2}$ edges in $N'(i)$. Thus, we obtain

$$\begin{aligned} \sum_{i \in V, j \in E} Z_{g,t}(x_i \cdot y_j) = vkZ_{g,t}\left(\frac{1+p}{\sqrt{2+2p}}\right) + \frac{vk\lambda}{2} Z_{g,t}\left(\frac{2p}{\sqrt{2+2p}}\right) \quad + v(v-k-1)\mu Z_{g,t}\left(\frac{p+q}{\sqrt{2+2p}}\right) \\ + \frac{v(v-k-1)(k-\mu)}{2} Z_{g,t}\left(\frac{2q}{\sqrt{2+2p}}\right) =: \Psi_B(v,k,\lambda,\mu,t). \end{aligned} \tag{3.4}$$

If $j \in E$ joins $x, y \in V$, we denote by n_j the number of edges in $N(x) \cap N(y)$. Clearly, $\sum_{j \in E} n_j = 6N$, where N is the number of 4-cliques in G. Fixing $j \in E$, considering various cases for $j' \in E$ and using Proposition 3.2, we obtain

$$\sum_{j,j'\in E} Z_{g,t}(y_j \cdot y_{j'}) = \sum_{j\in E}\Bigg(Z_{g,t}(1) + 2\lambda Z_{g,t}\left(\frac{1+3p}{2+2p}\right) + 2(k-\lambda-1)Z_{g,t}\left(\frac{1+2p+q}{2+2p}\right)$$

$$+ n_j Z_{g,t}\left(\frac{4p}{2+2p}\right) + 2(\lambda(\lambda-1)-2n_j)Z_{g,t}\left(\frac{3p+q}{2+2p}\right) + (\lambda(k-2\lambda)+2n_j)Z_{g,t}\left(\frac{2p+2q}{2+2p}\right)$$

$$+(((\mu-1)(k-\lambda-1)-\lambda(\lambda-1))+2n_j)Z_{g,t}\left(\frac{2p+2q}{2+2p}\right)$$

$$+2((k-\mu)(k-\lambda-1)-\lambda(k-2\lambda)-2n_j)Z_{g,t}\left(\frac{p+3q}{2+2p}\right) + (\lambda(k-2\lambda)+2n_j)Z_{g,t}\left(\frac{2p+2q}{2+2p}\right)$$

$$+\left(\frac{k(v-2k+\lambda)}{2} - (k-\mu)(k-\lambda-1) + \frac{\lambda(k-2\lambda)}{2} + n_j\right) Z_{g,t}\left(\frac{4q}{2+2p}\right)\Bigg)$$

$$= \Psi_{C_0}(v,k,\lambda,\mu,t) + N\Psi_{C_1}(v,k,\lambda,\mu,t), \tag{3.5}$$

where

$$\Psi_{C_0}(v,k,\lambda,\mu,t) := \frac{vk}{2}\Bigg(Z_{g,t}(1) + 2\lambda Z_{g,t}\left(\frac{1+3p}{2+2p}\right) + 2(k-\lambda-1)Z_{g,t}\left(\frac{1+2p+q}{2+2p}\right) \tag{3.6}$$

$$+2\lambda(\lambda-1)Z_{g,t}\left(\frac{3p+q}{2+2p}\right) + ((\mu-1)(k-\lambda-1)-\lambda(\lambda-1)+2\lambda(k-2\lambda))Z_{g,t}\left(\frac{2p+2q}{2+2p}\right)$$

$$+2((k-\mu)(k-\lambda-1)-\lambda(k-2\lambda))Z_{g,t}\left(\frac{p+3q}{2+2p}\right)$$

$$+\left(\frac{k(v-2k+\lambda)}{2} - (k-\mu)(k-\lambda-1) + \frac{\lambda(k-2\lambda)}{2}\right) Z_{g,t}\left(\frac{4q}{2+2p}\right)\Bigg)$$

and

$$\Psi_{C_1}(v,k,\lambda,\mu,t) := 6\sum_{l=0}^{4}(-1)^l\binom{4}{l}Z_{g,t}\left(\frac{(4-l)p+lq}{2+2p}\right). \tag{3.7}$$

Now we are ready to state and to prove our bound.

Theorem 3.3. *Let N be the number of 4-cliques in a (v,k,λ,μ) strongly regular graph. Then for any positive integer t one has*

$$(\Psi_B(v,k,\lambda,\mu,t))^2 \le \Psi_A(v,k,\lambda,\mu,t)\left(\Psi_{C_0}(v,k,\lambda,\mu,t) + N\Psi_{C_1}(v,k,\lambda,\mu,t)\right),$$

where Ψ_A, Ψ_B, Ψ_{C_0} and Ψ_{C_1} are defined in (3.3)*,* (3.4)*,* (3.6) *and* (3.7)*.*

Proof. If $G=(V,E)$ is the given graph, let $x_i \in \mathbb{R}^g$ be the Euclidean representation of $i \in V =: \mathcal{I}$ (satisfying (2.8)), and let $y_j := \frac{x_{j(1)}+x_{j(2)}}{\|x_{j(1)}+x_{j(2)}\|}$ for each edge $j \in E =: \mathcal{J}$, where j joins the vertices $j^{(1)}, j^{(2)} \in V$. The required estimate is exactly (3.1) with the notations introduced in (3.3), (3.4) and (3.5). □

For our applications, we choose $t = 4$. The resulting bound on N can be expressed in terms of a rational function of k, r, s of degree ≤ 10 in each variable (here r and s are the corresponding eigenvalues, see (2.3) and (2.5)). The expression for this rational function is quite lengthy and is provided in [BPR], where one can also find a table of non-trivial bounds on N for all admissible $v \leq 1300$.

The following is an immediate corollary of Theorem 3.3 needed for the proof of Theorem 1.1.

Corollary 3.4. *Any $SRG(76, 30, 8, 14)$ contains a K_4.*

Moreover, the bound from the Theorem 3.3 provides $N \geq \frac{2128}{55}$, so $N \geq 39$. In this case, in (3.3)–(3.7) we have $Z_{g,t}(\xi) = Z_{18,4}(\xi) = 54 - 2160\xi^2 + 7920\xi^4)$.

4. Tools and some computations

Throughout this section, let G be a (v, k, λ, μ) strongly regular graph.

4.1. **Relating frequencies of certain edge counts.**

Our first lemma relates the frequencies of degrees in an induced subgraph with frequencies of quantities of edges joining a vertex from outside the subgraph with vertices inside the subgraph.

Lemma 4.1. *Let H be a subgraph of G, $m = |H|$, define*

$$d_j := |\{x \in H : \text{there are exactly } j \text{ edges from } x \text{ to vertices in } H\}|$$

$$b_j := |\{x \in G \setminus H : \text{there are exactly } j \text{ edges from } x \text{ to vertices in } H\}|.$$

Then

$$\sum_{j\geq 0} b_j = v - m, \tag{4.1}$$

$$\sum_{j\geq 0} jb_j = mk - \sum_{j\geq 0} jd_j, \quad \text{and} \tag{4.2}$$

$$\sum_{j\geq 0} \binom{j}{2} b_j = \binom{m}{2}\mu - \sum_{j\geq 0} \binom{j}{2} d_j + \frac{1}{2}(\lambda - \mu) \sum_{j\geq 0} jd_j. \tag{4.3}$$

Proof. Counting the number of vertices in $G \setminus H$, we immediately get (4.1). For (4.2), consider the total number of edges from $G \setminus H$ to H. Finally, the left-hand-side of (4.3) is the number of paths of length 2 (say $x_1, x_2, x_3 \in G$) that originate and terminate in H ($x_1, x_3 \in H$) and go through $G \setminus H$ ($x_2 \in G \setminus H$). The number of paths x_1, x_2, x_3 with $x_1, x_3 \in H$ (and x_2 possibly in H) can be computed (using strong regularity) considering two cases: if x_1 and x_3 are adjacent, we obtain $\frac{\lambda}{2}\sum_{j\ge 0} jd_j$ paths, and if x_1 and x_3 are not adjacent, we have $\mu\left(\binom{m}{2} - \frac{1}{2}\sum_{j\ge 0} jd_j\right)$ paths. Subtracting the number of paths of length 2 that completely belong to H, which is $\sum_{j\ge 0}\binom{j}{2}d_j$, we obtain the desired result. □

We summarize the required applications of Lemma 4.1 in the following corollary, where for simplicity all unspecified entries of the sequences of frequencies are assumed to be zero.

Corollary 4.2. *Suppose* $(v, k, \lambda, \mu) = (76, 30, 8, 14)$. *With notations of Lemma 4.1, the following statements are valid.*

(i) If $m = 4$, $(d_j)_{j\ge 0} = (0, 0, 0, 4, \dots)$, *and* $b_4 = 0$, *then* $(b_j)_{j\ge 0} = (0, 36, 36, 0, \dots)$.

(ii) If $m = 6$, $(d_j)_{j\ge 0} = (0, 0, 0, 0, 6, \dots)$, *and* $b_0 = b_4 = b_5 = b_6 = 0$, *then* $(b_j)_{j\ge 0} = (0, 0, 54, 16, \dots)$.

(iii) If $m = 7$, $(d_j)_{j\ge 0} = (0, 0, 0, 0, 6, 0, 1, \dots)$, *and* $b_0 = b_5 = b_6 = b_7 = 0$, *then* $(b_j)_{j\ge 0} = (0, 0, 27, 42, 0, \dots)$.

(iv) If $m = 7$, $(d_j)_{j\ge 0} = (0, 0, 0, 0, 5, 2, \dots)$, *and* $b_0 = b_5 = b_6 = b_7 = 0$, *then* $(b_j)_{j\ge 0} = (0, 0, 28, 40, 1, \dots)$ *or* $(b_j)_{j\ge 0} = (0, 1, 25, 43, 0, \dots)$.

(v) If $m = 8$, $(d_j)_{j\ge 0} = (0, 0, 0, 0, 3, 4, 1, \dots)$, *and* $b_0 = b_1 = b_5 = b_6 = b_7 = b_8 = 0$, *then* $(b_j)_{j\ge 0} = (0, 0, 7, 56, 5, \dots)$.

(vi) If $m = 8$, $(d_j)_{j\ge 0} = (0, 0, 0, 0, 2, 6, \dots)$, *and* $b_0 = b_1 = b_5 = b_6 = b_7 = b_8 = 0$, *then* $(b_j)_{j\ge 0} = (0, 0, 8, 54, 6, \dots)$.

Proof. While we have only three linear equations and usually more than three non-zero unknowns (clearly $b_j = 0$ for $j > m$), we can utilize the fact that b_j are non-negative integers.

(i) We have $b_0 + b_1 + b_2 + b_3 = 72$, $b_1 + 2b_2 + 3b_3 = 108$, and $b_2 + 3b_3 = 36$. Subtracting the third equation from the second we get $b_1 + b_2 = 72$, then the first one gives $b_0 = b_3 = 0$, so consequently $b_1 = b_2 = 36$ from the last two equations.

(ii) As $b_1 + 2b_2 + 3b_3 = 156$ and $b_2 + 3b_3 = 102$, we get $b_1 + b_2 = 54$, so $b_1 + b_2 + b_3 = 70$ yields $b_3 = 16$. Back substitution gives $b_2 = 54$ and $b_1 = 0$.

(iii) Adding the equations $b_1+b_2+b_3+b_4 = 69$, $b_1+2b_2+3b_3+4b_4 = 180$, and $b_2+3b_3+6b_4 = 153$

with the coefficients 3, -2, and 1, respectively, we obtain $b_1 + b_4 = 0$, so $b_1 = b_4 = 0$, and the resulting system has only one solution $b_2 = 27$, $b_3 = 42$.
(iv) We obtain almost the same system as in (iii), the only difference is that 153 is replaced with 154. Hence, the same linear combination of the equations now provides $b_1+b_4 = 1$, yielding two possibilities: $(b_1, b_4) = (0, 1)$ or $(b_1, b_4) = (1, 0)$. Back substitution leads to two linear systems with unique solutions for b_2 and b_3 that are stated in the corollary.
(v), (vi) In these cases we have three unknowns and the system is non-degenerate, unique solutions as stated. □

Remark 4.3. Parts (i)–(iii) of Corollary 4.2 can be immediately obtained from a general result [Soi10, Theorem 1.2] on block intersection polynomials. The remaining cases (iv)–(vi) require a different treatment, so we included all the proofs above for completeness.

4.2. **Sums of vectors for vertex partitions.** Let $\pi = \{G_1, \dots, G_l\}$ be a partition of a subset $\widetilde{V} \subset V$ of the vertices of the graph $G = (V, E)$ (which, recall, is a (v, k, λ, μ) strongly regular graph throughout this section). Following Section 2.2, we can use the Euclidean representation of G in $\mathbb{R}^g$ (where g is defined in (2.6)) to obtain vectors $X_j := \sum_{i \in G_j} x_i$, $j = 1, \dots, l$. Denote by $M(\pi, p, q)$ the Gram matrix of X_j, i.e., $M_{i,j} := X_i \cdot X_j$, $i, j = 1, \dots, l$, where p and q are from (2.8). The following lemma is straightforward from (2.7).

Lemma 4.4. *If π has edge matrix $\mathcal{E}_\pi = (a_{i,j})_{i,j=1}^l$, and $m_j = |G_j|$, then the entries of $M(\pi, p, q)$ can be computed as follows:*

$$M_{i,i} = m_i + 2a_{i,i}p + (m_i(m_i - 1) - 2a_{i,i})q, \quad M_{i,j} = a_{i,j}p + (m_i m_j - a_{i,j})q.$$

As M is non-negative definite, its determinant is non-negative. We will use this fact very frequently, so for convenience we state it in the following lemma.

Lemma 4.5. $\det M(\pi, p, q) \geq 0$.

Let us illustrate an immediate consequence for our particular graph. By K_l we denote a complete graph on l vertices.

Corollary 4.6. *Suppose $(v, k, \lambda, \mu) = (76, 30, 8, 14)$. Then G does not contain K_5 as a subgraph.*

Proof. If G' are the vertices of K_5, then $M = M(\{G'\}, -\frac{4}{15}, \frac{7}{45}) = (-\frac{1}{3})$ and $\det M < 0$. □

This corollary also follows (and, in fact, it is the same proof expressed in a slightly different language) from the Hoffman bound on the independence number applied to the complement of G, see, e.g. [BH12, Theorem 3.5.2].

Using Corollary 4.6 and Corollary 4.2 (i), we will obtain a stronger statement during the proof of Lemma 5.1, which we state here for convenience.

Proposition 4.7. *Suppose* $(v,k,\lambda,\mu) = (76,30,8,14)$. *Then* G *does not contain either* K_5 *or* $K_5 - e$ *as a subgraph, where* $K_5 - e$ *denotes a* K_5 *with one edge removed.*

If $\det M(\pi,p,q) = 0$, the system of vectors X_j is linearly dependent, and we can derive even more information.

Lemma 4.8. *Suppose that for some real* λ_j, $j = 1,\dots,l$, *we have* $\sum_j \lambda_j X_j = 0$. *For any vertex* $z \in G$, *let* e_j *be the number of neighbors of* z *in* G_j. *Then*

$$\sum_{j: z\notin G_j} \lambda_j(pe_j + q(|G_j| - e_j)) + \sum_{j: z\in G_j} \lambda_j(1 + pe_j + q(|G_j| - 1 - e_j)) = 0. \tag{4.4}$$

Proof. Follows directly from $x_z \cdot (\sum_j \lambda_j X_j) = 0$. □

Remark 4.9. If $z \in G_j$, then $p\tilde{e}_j + q(|G_j| - \tilde{e}_j) = 1 + pe_j + q(|G_j| - 1 - e_j)$ for $\tilde{e}_j = \frac{1-q}{p-q} + e_j$. Therefore, with

$$\tilde{e}_j := \begin{cases} e_j, & \text{if } z \notin G_j, \\ \frac{1-q}{p-q} + e_j, & \text{if } z \in G_j, \end{cases}$$

we can rewrite (4.4) as

$$\sum_j \lambda_j(p\tilde{e}_j + q(|G_j| - \tilde{e}_j)) = 0.$$

Such unified form of (4.4) is somewhat more convenient for implementation in a computer algebra system and is utilized in our SAGE worksheets [BPR].

Remark 4.10. Lemmas 4.4, 4.5, and 4.8 were stated for Euclidean representation of G in $\mathbb{R}^g$. The same results are valid for Euclidean representation of G in $\mathbb{R}^f$ (obtained by considering the complement of G), in which case the values of p and q will change (see Remark 2.2 and (2.8)), which may lead to different (sometimes more useful) conclusions.

4.3. **Projections.** Understanding structure of systems of point sets on Euclidean sphere may be a challenging task. It becomes easier if we are working in a space with small dimension, such as $\mathbb{R}^2$ or $\mathbb{R}^3$. A natural way to reduce the problem to such settings is to consider orthogonal projections.

Recall that following Section 2.2, we are considering the Euclidean representation of a (v, k, λ, μ) strongly regular graph G in $\mathbb{R}^g$ which assigns the unit vectors $x_i \in \mathbb{R}^g$ to each $i \in V$, satisfying (2.7) and other properties from Section 2.2. For any vertex $j \in V$, we want to compute the projection x'_j of x_j onto the linear subspace spanned by $\{x_i, i \in \widetilde{G}\}$ for some subgraph $\widetilde{G} \subset G$. Clearly, x'_j is a linear combination of $\{x_i, i \in \widetilde{G}\}$, and the orthogonality conditions $(x'_j - x_j) \cdot x_t = 0$, $t \in \widetilde{G}$, can be used to form a linear system of equations on the coefficients of the linear combination. Namely, if

$$x'_j = \sum_{i \in \widetilde{G}} \alpha_i x_i, \tag{4.5}$$

where, of course, α_i also depends on j and $\widetilde{G}$, then

$$x_j \cdot x_t = \sum_{i \in \widetilde{G}} \alpha_i x_i \cdot x_t, \quad t \in \widetilde{G}. \tag{4.6}$$

All the dot products in this system can be computed by (2.7) if we know the adjacency matrix of $\widetilde{G} \cup \{x_j\}$, and then, if the matrix turns out to be non-degenerate, we can compute the coefficients α_i, $i \in \widetilde{G}$.

In practice, we will use the above computations in a special situation.

Proposition 4.11. *Suppose $\pi = \{G_1, \dots, G_l\}$ is an equitable partition of $\widetilde{G}$ with degree matrix $\mathcal{D}_\pi = (b_{w,u})$ (see Section 2.3). Further assume that for every u, $1 \le u \le l$, j is either adjacent to each vertex of G_u or disjoint with each vertex of G_u. Then in (4.5) we can take $\alpha_i = \alpha_u$ for $i \in G_u$, $1 \le u \le l$, and (4.6) becomes*

$$q + (p - q)a_w = (1 - q)\alpha_w + \sum_{u=1}^{l} \alpha_u \big(p b_{w,u} + q(|G_u| - b_{w,u})\big), \quad 1 \le w \le l, \tag{4.7}$$

where $a_w \in \{0, 1\}$ is the number of edges from j to any vertex of G_w.

Proof. If $u \ne w$, then for $t \in G_w$ there are exactly $b_{w,u}$ neighbors of t in G_u, so

$$\sum_{i \in G_u} x_i \cdot x_t = p b_{w,u} + q(|G_u| - b_{w,u}).$$

If $u = w$, the for $t \in G_w$ there are exactly $b_{w,w}$ neighbors of t in G_w, and we have to account for t itself, so

$$\sum_{i \in G_w} x_i \cdot x_t = 1 + pb_{w,w} + q(|G_w| - b_{w,w} - 1) = pb_{w,w} + q(|G_w| - b_{w,w}) + (1 - q).$$

□

Our next goal is to compute the dot product of the projections.

Proposition 4.12. *Suppose $x'_{j^{(1)}}$ and $x'_{j^{(2)}}$ are projections of $x_{j^{(1)}}$ and $x_{j^{(2)}}$ onto the linear span of $\{x_i, i \in \widetilde{G}\}$, where $\widetilde{G} \subset G$ has two corresponding equitable partitions $\pi^{(1)} = \{G_1^{(1)}, \dots, G_l^{(1)}\}$ and $\pi^{(2)} = \{G_1^{(2)}, \dots, G_l^{(2)}\}$ each satisfying the conditions of Proposition 4.11. Suppose further that both systems (4.7) admit the same solution $\alpha_u = \alpha_u^{(1)} = \alpha_u^{(2)}$, $1 \le u \le l$. Then*

$$x'_{j^{(1)}} \cdot x'_{j^{(2)}} = \sum_{(u,w)} \alpha_u \alpha_w \big(|G_{u,w}| + 2e_{u,w}p + (|G_{u,w}|(|G_{u,w}| - 1) - 2e_{u,w})q)\big) + \sum_{(u,w) \ne (\tilde{u},\tilde{w})} \alpha_u \alpha_{\tilde{w}} \big(e_{u,w,\tilde{u},\tilde{w}}p + (|G_{u,w}||G_{\tilde{u},\tilde{w}}| - e_{u,w,\tilde{u},\tilde{w}})q\big) \tag{4.8}$$

where $G_{u,w} = G_u^{(1)} \cap G_w^{(2)}$, $e_{u,w}$ is the number of edges in $G_{u,w}$, and $e_{u,w,\tilde{u},\tilde{w}}$ is the number of edges between $G_{u,w}$ and $G_{\tilde{u},\tilde{w}}$ for $(u,w) \ne (\tilde{u},\tilde{w})$. The first summation in (4.8) is taken over all l^2 possible values of u and w; and the second summation is over all $l^4 - l^2$ values of u, w, $\tilde{u}$, and $\tilde{w}$, satisfying $(u,w) \ne (\tilde{u},\tilde{w})$.

Proof. From (4.5) we have

$$x'_{j^{(1)}} = \sum_{u,w} \alpha_u \sum_{i \in G_{u,w}} x_i \quad \text{and} \quad x'_{j^{(2)}} = \sum_{u,w} \alpha_w \sum_{i \in G_{u,w}} x_i.$$

Using (2.7) (as in Lemma 4.4)

$$\Big(\alpha_u \sum_{i \in G_{u,w}} x_i\Big) \cdot \Big(\alpha_w \sum_{i \in G_{u,w}} x_i\Big) = \alpha_u \alpha_w \big(|G_{u,w}| + 2e_{u,w}p + (|G_{u,w}|(|G_{u,w}| - 1) - 2e_{u,w})q)\big),$$

while for $(u,w) \ne (\tilde{u},\tilde{w})$

$$\Big(\alpha_u \sum_{i \in G_{u,w}} x_i\Big) \cdot \Big(\alpha_{\tilde{w}} \sum_{i \in G_{\tilde{u},\tilde{w}}} x_i\Big) = \alpha_u \alpha_{\tilde{w}} \big(e_{u,w,\tilde{u},\tilde{w}}p + (|G_{u,w}||G_{\tilde{u},\tilde{w}}| - e_{u,w,\tilde{u},\tilde{w}})q\big),$$

and (4.8) follows. □

In practical applications of Proposition 4.12, we will often have that the numbers of edges between different components of $\widetilde{G}$ can be computed in terms of only a few unknowns, some of which will cancel after simplifying the sums in (4.8), so the resulting formula for the dot product of the projections will be short. Namely, we will obtain that

$$x'_{j^{(1)}} \cdot x'_{j^{(2)}} = \alpha n_{j^{(1)},j^{(2)}} + \beta \tag{4.9}$$

for some constants α and β and a certain quantity $n_{j^{(1)},j^{(2)}}$ depending on the vertices $j^{(1)}$ and $j^{(2)}$. Now we record a simple straightforward computation in the space orthogonal to the linear span of $\{x_i, i \in \widetilde{G}\}$.

Proposition 4.13. *If under the assumptions of Proposition 4.12, the equation* (4.9) *holds with* $n_{j^{(1)},j^{(1)}} = n_{j^{(2)},j^{(2)}}$, *then for the orthogonal components*

$$x''_{j^{(1)}} := x_{j^{(1)}} - x'_{j^{(1)}} \quad \textit{and} \quad x''_{j^{(2)}} := x_{j^{(2)}} - x'_{j^{(2)}},$$

we have $\|x''_{j^{(1)}}\| = \|x''_{j^{(2)}}\|$, *and the cosine of the angle between* $x''_{j^{(1)}}$ *and* $x''_{j^{(2)}}$ *is*

$$\frac{x''_{j^{(1)}} \cdot x''_{j^{(2)}}}{\|x''_{j^{(1)}}\|\|x''_{j^{(2)}}\|} = \frac{x_{j^{(1)}} \cdot x_{j^{(2)}} - (\alpha n_{j^{(1)},j^{(2)}} + \beta)}{1 - (\alpha n_{j^{(1)},j^{(1)}} + \beta)}. \tag{4.10}$$

4.4. **Rank computation.** Throughout this subsection, let G be a $(76, 30, 8, 14)$ strongly regular graph . Next series of lemmas is devoted to computations of ranks of certain subspaces generated by linear combinations of vectors from the Euclidean representation. For a subgraph $\widetilde{G} \subset G$ we denote by $B(\widetilde{G})$ the Gram matrix $(x_i \cdot x_j)_{i,j \in \widetilde{G}}$. By Proposition 2.1, $\operatorname{rank}(\operatorname{lin}(\{x_i,\ i \in \widetilde{G}\})) = \operatorname{rank}(B(\widetilde{G}))$. If A is the adjacency matrix of $\widetilde{G}$, then $B(\widetilde{G}) = pA + I + q(J - I - A)$ by (2.7).

Lemma 4.14. *If* $\widetilde{G}$ *is a* 16*-coclique, then* $\operatorname{rank}(B(\widetilde{G})) = 16$*. If* $\widetilde{G}$ *is a* $K_{6,10}$*, then* $\operatorname{rank}(B(\widetilde{G})) = 15$.

Proof. We have $p = -\frac{4}{15}$ and $q = \frac{7}{45}$, so $B(\widetilde{G})$ can be explicitly written. Computing the dimension of the kernels of these matrices is an easy linear algebra exercise. We obtain that the matrix for 16-coclique is non-degenerate, and the matrix for $K_{6,10}$ has one-dimensional kernel. □

We also need to use certain spectral arguments to compute the rank in other situations. A key observation is that if $\boldsymbol{e} := (1, 1, \dots, 1)$ is an eigenvector of the adjacency matrix A, then any eigenvector $\boldsymbol{v}$ corresponding to a different eigenvalue is orthogonal to $\boldsymbol{e}$, and hence $J\boldsymbol{v}$ is the zero vector.

Lemma 4.15. *If* $\widetilde{G}$ *is a* $(40, 12, 2, 4)$ *strongly regular graph, then* $\operatorname{rank}(B(\widetilde{G})) = 16$.

Proof. If A is the adjacency matrix of $\widetilde{G}$, it has eigenvalue 12 of multiplicity 1 with eigenvector $\boldsymbol{e}$, eigenvalue 2 of multiplicity 24, and eigenvalue -4 of multiplicity 15 (by (2.3)–(2.6)). We notice that $\boldsymbol{e}$ is the eigenvector of $B(\widetilde{G}) = -\frac{4}{15}A + I + \frac{7}{45}(J - I - A)$ with eigenvalue 2:

$$B(\widetilde{G})\boldsymbol{e} = \left(-\frac{4}{15}12 + 1 + \frac{7}{45}(40 - 1 - 12)\right)\boldsymbol{e} = 2\boldsymbol{e}.$$

Any eigenvector $\boldsymbol{v}$ of A corresponding to the eigenvalue 2 will be an eigenvector for $B(\widetilde{G})$ with eigenvalue 0 (here we use the observation that $\boldsymbol{v}$ is orthogonal to $\boldsymbol{e}$, hence $J\boldsymbol{v}$ is zero):

$$B(\widetilde{G})\boldsymbol{v} = \left(-\frac{4}{15}2 + 1 + \frac{7}{45}(0 - 1 - 2)\right)\boldsymbol{v} = 0\boldsymbol{v},$$

and the dimension of this eigenspace is 24. Finally, if $\boldsymbol{v}$ is an eigenvector of A for the eigenvalue -4, then $\boldsymbol{v}$ is an eigenvector for $B(\widetilde{G})$ with eigenvalue $\frac{38}{15}$:

$$B(\widetilde{G})\boldsymbol{v} = \left(-\frac{4}{15}(-4) + 1 + \frac{7}{45}(0 - 1 - (-4))\right)\boldsymbol{v} = \frac{38}{15}\boldsymbol{v},$$

and the multiplicity is 15. The sum of the multiplicities of non-zero eigenvalues is 16. □

By C_l we denote the (undirected) l-cycle. The spectrum of C_l is $\{2\cos(2\pi j/l)\}_{j=1}^{l}$ (see, e.g. [BH12, Section 1.4.3]). Clearly, the spectrum of a disjoint union of cycles will be the union of the spectra.

Lemma 4.16. *If* $\widetilde{G}$ *is a disjoint union of* n *cycles with* 20 *vertices in total, then* $\operatorname{rank}(B(\widetilde{G})) = 21 - n$.

Proof. If A is the adjacency matrix of $\widetilde{G}$, then $\boldsymbol{e}$ is the eigenvector of A corresponding to the eigenvalue 2. The multiplicity of this eigenvalue is n by the preceding discussion on the structure of the spectrum of $\widetilde{G}$. Observe that $\boldsymbol{e}$ is an eigenvector for $B(\widetilde{G})$ with eigenvalue $\frac{49}{15}$:

$$B(\widetilde{G})\boldsymbol{e} = \left(-\frac{4}{15}2 + 1 + \frac{7}{45}(20 - 1 - 2)\right)\boldsymbol{e} = \frac{49}{15}\boldsymbol{e}.$$

We can choose $n - 1$ linearly independent eigenvectors of A corresponding to the eigenvalue 2 so that each such vector $\boldsymbol{v}$ is orthogonal to $\boldsymbol{e}$. Then $\boldsymbol{v}$ is an eigenvector of $B(\widetilde{G})$ with eigenvalue zero:

$$B(\widetilde{G})\boldsymbol{v} = \left(-\frac{4}{15}2 + 1 + \frac{7}{45}(0 - 1 - 2)\right)\boldsymbol{v} = 0\boldsymbol{v}.$$

It remains to show that any eigenvector $\boldsymbol{v}$ of A corresponding to an eigenvalue $\tilde{\lambda} \neq 2$ is an eigenvector of $B(\widetilde{G})$ with a non-zero eigenvalue. This is straightforward:

$$B(\widetilde{G})\boldsymbol{v} = \left(-\frac{4}{15}\tilde{\lambda} + 1 + \frac{7}{45}(0 - 1 - \tilde{\lambda})\right)\boldsymbol{v} = -\frac{19}{45}(\tilde{\lambda} - 2)\boldsymbol{v}.$$

So zero is an eigenvalue of $B(\widetilde{G})$ of multiplicity $(n-1)$, hence $\operatorname{rank}(B(\widetilde{G})) = 20 - (n-1) = 21 - n$. □

5. Reduction to $SRG(40, 12, 2, 4)$ or 16-coclique or $K_{6,10}$ as a subgraph

Theorem 1.1 follows immediately from the following four lemmas. Recall that $N(z)$ and $N'(z)$ are the sets of neighbors and non-neighbors of a vertex z, respectively.

Lemma 5.1. *If G is a $SRG(76, 30, 8, 14)$, then there is a subgraph $\widetilde{G}$ of G satisfying one of the following statements:*
(i) $\widetilde{G}$ is a $SRG(40, 12, 2, 4)$, and for any $z \in G \setminus \widetilde{G}$ both $N(z) \cap \widetilde{G}$ and $N'(z) \cap \widetilde{G}$ are 4-regular subgraphs on 20 vertices, and $|N(z_1) \cap N(z_2) \cap \widetilde{G}| = 8$ for any adjacent $z_1, z_2 \in G \setminus \widetilde{G}$; or
(ii) $\widetilde{G}$ is a 16-coclique; or
(iii) $\widetilde{G}$ is a $K_{6,10}$.

Recall that n-coclique is a graph with n vertices without edges, and $K_{m,n}$ is the complete bipartite graph with two components with m and n vertices respectively.

Lemma 5.2. *If G is a $SRG(76, 30, 8, 14)$, there cannot be an induced subgraph $\widetilde{G} \subset G$ which is a $SRG(40, 12, 2, 4)$, and, in addition, for any $z \in G \setminus \widetilde{G}$ both $N(z) \cap \widetilde{G}$ and $N'(z) \cap \widetilde{G}$ are 4-regular subgraphs on 20 vertices, and $|N(z_1) \cap N(z_2) \cap \widetilde{G}| = 8$ for any adjacent $z_1, z_2 \in G \setminus \widetilde{G}$.*

Lemma 5.3. *16-coclique cannot be an induced subgraph of $SRG(76, 30, 8, 14)$.*

Lemma 5.4. *$K_{6,10}$ cannot be an induced subgraph of $SRG(76, 30, 8, 14)$.*

In this section we will prove Lemma 5.1 only.

Proof of Lemma 5.1. Let G be a $(76, 30, 8, 14)$ strongly regular graph, $G_0 \subset G$ be the vertices of K_4 existing due to Corollary 3.4. Apply Lemma 4.1 with $H = G_0$, noting that by

Corollary 4.6 we have $d_j = 0$ with $j \geq 5$. More specifically, by Corollary 4.2 (i), we get $(b_j)_{j\geq 0} = (0, 36, 36, 0, \dots)$.

The above argument can be used to establish Proposition 4.7. Indeed, as the choice of K_4 is arbitrary, since $b_3 = b_4 = 0$, we immediately obtain Proposition 4.7 selecting as K_4 arbitrary four vertices of $K_5 - e$ [[not containing the missing edge]]. Now we come back to our Lemma.

For $j = 1, 2$, we define by G_j the subgraph of $G \setminus G_0$ with vertices connected to exactly j vertices of G_0. Note that G is partitioned into G_0, G_1, and G_2.

Next we claim that $\{G_0, G_1, G_2\}$ is an equitable partition of G with degree matrix

$$\mathcal{D}_{\{G_0,G_1,G_2\}} = \begin{pmatrix} 3 & 1 & 2 \\ 9 & 11 & 18 \\ 18 & 18 & 10 \end{pmatrix}. \tag{5.1}$$

(In particular, G_1 and G_2 are regular graphs with regularity 11 and 10 respectively.) The first row is already established, the rest is an easy consequence of strong regularity with very similar calculations in each case. For example, we will illustrate how to obtain 10-regularity of G_2. Take a vertex $x \in G_2$, let t be the number of neighbors of x in G_2. Since G is 30-regular, there are $28 - t$ neighbors of x in G_1 (two in G_0). Calculating the number of paths of length 2 that start at x and end in G_0 in two ways, we get

$$2 \cdot 3 + (28 - t) \cdot 1 + t \cdot 2 = 2 \cdot 8 + 2 \cdot 14,$$

which yields $t = 10$, as required. Terms in the left hand side correspond to paths with middle vertex in G_0, G_1, G_2, respectively. Terms in the right hand side correspond to the cases whether the terminal vertex from G_0 is connected to x or not, and use parameters of strong regularity of G.

Now we consider two cases depending on whether G_2 contains a triangle.

Case 1. G_2 has no triangles. Then we will show that $\widetilde{G} := G \setminus G_2$ is $SRG(40, 12, 2, 4)$. For any vertex $x \in \widetilde{G}$ let $H_x \subset G_2$ be the vertices adjacent to x. By (5.1), we always have $|H_x| = 18$. As G_2 has no triangles, strong regularity of G provides that each edge of G_2 belongs to exactly 8 triangles, where all 8 third vertices belong to $\widetilde{G}$. Therefore, the average number of edges in 18-vertex subgraphs H_x over all $x \in \widetilde{G}$ is precisely $\frac{180 \cdot 8}{40} = 36$ (here 180 is the number of edges in G_2 and $40 = |\widetilde{G}|$). Let w be the number of edges in H_x for some fixed $x \in \widetilde{G}$. If $x \notin G_0$, we will use Lemma 4.5 with $\pi = \{G_0, H_x, \{x\}\}$. In notations of Lemma 4.4, we have the cardinalities $(m_1, m_2, m_3) = (4, 18, 1)$ and the edge matrix

$$\mathcal{E}_\pi = \begin{pmatrix} 6 & 2 \cdot 18 & 1 \\ & w & 18 \\ & & 0 \end{pmatrix}.$$

Now by Lemmas 4.4 and 4.5 and straightforward computations (see the supplied SAGE worksheet for these and subsequent computations), we obtain $\det M(\pi, p, q) = \frac{722}{1125}(36 - w) \geq 0$, so $w \leq 36$. As $x \in \widetilde{G}$ was arbitrary, due to the above computation of the average value of w, we obtain $w = 2|H_x| = 36$.

For any $x \in \widetilde{G}$ with $\pi = \{G_0, H_x, \{x\}\}$ we have $\det M(\pi, p, q) = 0$ and using notations of Lemma 4.8 we can set $(\lambda_1, \lambda_2, \lambda_3) = (1, 1/4, 1)$. If $z \in G_1$ is adjacent to x, then $e_1 = e_3 = 1$, so by Lemma 4.8, $e_2 = 6$. If $z \in G_1$ is not adjacent to x, then $e_1 = 1$, $e_3 = 0$, and by Lemma 4.8, $e_2 = 10$. Similar computations can be performed if $z \in G_0$ also using Lemma 4.8, leading to the same conclusion about e_2. In summary, for any $z \in \widetilde{G}$, the number of neighbors of z in H_x is equal to 6 or 10 when z is or is not adjacent to x, respectively. Therefore, any pair of adjacent vertices in $\widetilde{G}$ has 8 common neighbors in G, 6 of which are in G_2, therefore exactly $8 - 6 = 2$ are in $\widetilde{G}$. Similarly, any pair of non-adjacent vertices in $\widetilde{G}$ has exactly $14 - 10 = 4$ neighbors in $\widetilde{G}$. It readily follows from (5.1) that $\widetilde{G}$ is a regular graph of degree 12, so $\widetilde{G}$ is $SRG(40, 12, 2, 4)$.

To complete Case 1, it remains to show that for any $z \in G_2$ both $N(z) \cap \widetilde{G}$ and $N'(z) \cap \widetilde{G}$ are 4-regular subgraphs on 20 vertices, and that $|N(z_1) \cap N(z_2) \cap \widetilde{G}| = 8$ for any adjacent $z_1, z_2 \in G_2$.

Indeed, $|N(z)| = 30$, and by (5.1) G_2 is 10-regular, so $|N(z) \cap \widetilde{G}| = |N'(z) \cap \widetilde{G}| = 20$. As G_2 has no triangles, $N(z) \cap G_2$ is a 10-coclique, therefore, as $N(z)$ is 8-regular, there are exactly $8 \cdot 10 = 80$ edges between $N(z) \cap G_2$ and $N(z) \cap \widetilde{G}$. Moreover, this implies that there are exactly $120 - 80 = 40$ edges in $N(z) \cap \widetilde{G}$. Now we claim that $N(z) \cap \widetilde{G}$ is regular, which would imply the 4-regularity as we know the total edge count in this subgraph on 20 vertices to be 40. Let $x_i \in \mathbb{R}^{15}$ be the Euclidean representation of vertex $i \in \widetilde{G}$, which is $SRG(40, 12, 2, 4)$, and then (2.7) holds with $p = \frac{1}{6}$ and $q = -\frac{1}{9}$. Taking $X := \sum_{i \in N(z) \cap \widetilde{G}} x_i$, we obtain

$$X \cdot X = 20 + 80\frac{1}{6} - 300\frac{1}{9} = 0,$$

so $X \cdot x_i = 0$ for any $i \in N(z) \cap \widetilde{G}$, implying regularity. Each vertex $i \in N(z) \cap \widetilde{G}$ has 4 neighbors in $N(z) \cap \widetilde{G}$, hence 8 neighbors in $N'(z) \cap \widetilde{G}$. This directly leads to the fact that $N'(z) \cap \widetilde{G}$ also has 40 edges, hence, it is 4-regular by the same arguments.

Finally, if $z_1, z_2 \in G_2$ are adjacent, then since G_2 has no triangles, all 8 common neighbors of z_1 and z_2 are in $\widetilde{G}$.

Case 2. G_2 has a triangle G_3. There will be several subcases depending on how G_0 is connected with G_3. For each vertex in G_0, we consider the number of its neighbors in G_3, and record the resulting 4-tuple in descending order. We classify the subcases using such 4-tuples. As $G_3 \subset G_2$, each vertex of G_3 is connected to exactly two vertices of G_0, so the sum of the entries of such 4-tuples is always 6, and each entry does not exceed 3. We consider the cases in the reverse lexicographical order.

The subcase $(3,3,0,0)$ is impossible due to Corollary 4.6 as the two vertices that have three neighbors in G_3 would form a K_5 subgraph together with G_3. Similarly, $(3,2,1,0)$ is impossible due to Proposition 4.7 as we can find a $K_5 - e$ as a subgraph.

All of the remaining subcases begin with application of Lemma 4.1 for a certain H either for $H = G_0 \cup G_3$ or for a closely related graph. We will use the notations of that Lemma, in particular, $(d_j)_{j\geq 0}$ is the sequence of frequencies of degrees in the subgraph H and $(b_j)_{j\geq 0}$ is the sequence of frequencies of numbers of neighbors from H for the vertices in $G \setminus H$.

Subcase $(3,1,1,1)$. Take $H = G_0 \cup G_3$. We have $(d_j)_{j\geq 0} = (0,0,0,0,6,0,1,\dots)$. Each vertex in $G \setminus H$ is either in G_1 or G_2, so it has either one or two neighbors in G_0. Therefore, $b_0 = 0$ and $b_6 = b_7 = 0$. Next we claim that $b_5 = 0$. Indeed, otherwise, consider a vertex $x \in G \setminus H$ connected to five vertices from H, two must be from G_0 and three from G_3. If y is the vertex of G_0 connected to all three vertices from G_3, then x, y and G_3 form either a K_5, or a $K_5 - e$, so we get a contradiction by Proposition 4.7. Now we can apply Corollary 4.2 (iii), and get $(b_j)_{j\geq 0} = (0,0,27,42,0,\dots)$.

As above, let $y \in G_0$ be the vertex of degree 6 in H, so y is connected with every other vertex of H, moreover, every other vertex has degree 4 in H. There are $30 - 6 = 24$ vertices of $G \setminus H$ connected to y, let G_4 be the set of such vertices. We write $G_4 = G_5 \cup G_6$, where G_5 is the set of vertices with 2 neighbors in H (1 neighbor in $H \setminus \{y\}$) and G_6 is the set of vertices with 3 neighbors in H (2 neighbors in $H \setminus \{y\}$). We have $|G_5| + |G_6| = 24$. Since $G_4 \cup H \setminus \{y\}$ is 8-regular, each vertex of $H \setminus \{y\}$ has 5 neighbors in G_4. Counting the edges between G_4 and $H \setminus \{y\}$, we obtain $|G_5| + 2|G_6| = 5 \cdot 6$, so $|G_5| = 18$ and $|G_6| = 6$.

Let w be the number of edges in G_6. Take $\pi = \{G_6, H \setminus \{y\}, \{y\}\}$ and use Lemma 4.5. In notations of Lemma 4.4, we have the cardinalities $(m_1, m_2, m_3) = (6, 6, 1)$ and the edge matrix

$$\mathcal{E}_\pi = \begin{pmatrix} w & 12 & 6 \\ & 9 & 6 \\ & & 0 \end{pmatrix}.$$

Now by Lemmas 4.4 and 4.5, we obtain $\det M(\pi, p, q) = -\frac{1444}{3375} w \geq 0$, so $w = 0$. Therefore, $\det M(\pi, p, q) = 0$, and computing the kernel of $M(\pi, p, q)$, we can set $(\lambda_1, \lambda_2, \lambda_3) = (1, 4, 8)$ in notations of Lemma 4.8. Let G_7 be the set of $9 = b_2 - |G_5| = 27 - 18$ vertices from $G \setminus H$ having exactly 2 neighbors in H. We apply Lemma 4.8 for any $z \in G_7$. Obviously, z is not adjacent to y and has exactly 2 neighbors in $H \setminus \{y\}$, so $e_2 = 2$ and $e_3 = 0$, then by Lemma 4.8 $e_1 = 6$. This means that any vertex of G_7 is adjacent to any vertex of G_6. Clearly, y is adjacent to all vertices of G_6 and not adjacent to any of the vertices of G_7. To establish that the subgraph $G_6 \cup G_7 \cup \{y\}$ is $K_{6,10}$, it remains to show that there are no edges in $G_7 \cup \{y\}$. Take $\pi = \{G_7 \cup \{y\}, G_6\}$, let w be the number of edges in $G_7 \cup \{y\}$, then by Lemmas 4.4 and 4.5, $\det M(\pi, p, q) = -\frac{1216}{135} w \geq 0$, therefore $w = 0$, which completes the proof for the subcase $(3, 1, 1, 1)$ exhibiting a $K_{6,10}$ subgraph.

Subcase $(2, 2, 2, 0)$**.** Let G_8 be the set of the three vertices of G_0 with degrees 5 in $G_0 \cup G_3$. We take $H = G_8 \cup G_3$. Note that H is 4-regular on 6 vertices. We will show that if G contains such H as a subgraph, then there is a 16-coclique in G. We have $(d_j)_{j \geq 0} = (0, 0, 0, 0, 6, 0, \dots)$. For any $x \in G \setminus H$, take $\pi = \{H, \{x\}\}$, let w be the number of neighbors of x in H. Then $\mathcal{E}_\pi = \left(\begin{smallmatrix} 12 & w \\ & 0 \end{smallmatrix}\right)$, and by Lemmas 4.4 and 4.5, $\det M(\pi, p, q) = -\frac{1}{2025}(19w - 42)^2 + \frac{8}{15} \geq 0$, providing $1 \leq w \leq 3$, i.e., $b_0 = b_4 = b_5 = \dots = 0$. By Corollary 4.2 (ii), $(b_j)_{j \geq 0} = (0, 0, 54, 16, \dots)$. Let G_9 be the set of 16 vertices having 3 neighbors in H. Take $\pi = \{G_9, H\}$, let w be the number of edges in G_9. Then $\mathcal{E}_\pi = \left(\begin{smallmatrix} w & 48 \\ & 12 \end{smallmatrix}\right)$, and by Lemmas 4.4 and 4.5, $\det M(\pi, p, q) = -\frac{304}{675} w \geq 0$, so $w = 0$, and G_9 is the required subgraph.

Subcase $(2, 2, 1, 1)$**.** Take $H = G_0 \cup G_3$. We have $(d_j)_{j \geq 0} = (0, 0, 0, 0, 5, 2, \dots)$. For any $x \in G \setminus H$, take $\pi = \{H, \{x\}\}$, let w be the number of neighbors of x in H. Then $\mathcal{E}_\pi = \left(\begin{smallmatrix} 15 & w \\ & 0 \end{smallmatrix}\right)$, and by Lemmas 4.4 and 4.5, $\det M(\pi, p, q) = -\frac{1}{2025}(19w - 42)^2 + \frac{13}{15} \geq 0$, so $1 \leq w \leq 4$, i.e., $b_0 = b_5 = b_6 = \dots = 0$. By Corollary 4.2 (iv), either $(b_j)_{j \geq 0} = (0, 0, 28, 40, 1, \dots)$ or $(b_j)_{j \geq 0} = (0, 1, 25, 43, 0, \dots)$.

If $(b_j)_{j\geq 0} = (0, 0, 28, 40, 1, \dots)$, let $y \in G \setminus H$ be the vertex with exactly four neighbors in H. There are two possibilities: either $y \in G_1$ or $y \in G_2$.

Suppose $y \in G_1$. For any $x \in G_2 \setminus G_3$, there are exactly two neighbors in G_0, and as $b_5 = 0$ and $b_4 = 1$ while $y \neq x$, there is at most one neighbor of x in G_3. Let G_{10} be the set of such $x \in G_2 \setminus G_3$ having no neighbors in G_3, and G_{11} be the set of $x \in G_2 \setminus G_3$ with exactly one neighbor in G_3. Recall that G_2 is 10-regular and $G_3 \subset G_2$. Therefore, each vertex of G_3 has exactly 8 neighbors in G_{11}, and $|G_{11}| = 24$. Hence, $|G_{10}| = |G_2| - |G_3| - |G_{11}| = 36 - 3 - 24 = 9$. Further, by (5.1), y has 18 neighbors in G_2, so at least $18 - |G_3| - |G_{10}| = 6$ of them belong to G_{11}. Let G_{12} be any 6 vertices from G_{11} connected to y. We claim that $G_{12} \cup G_{10} \cup \{y\}$ is $K_{6,10}$. Take $\pi = \{G_{12}, G_3, G_0, \{y\}\}$, let w be the number of edges in G_{12}. In notations of Lemma 4.4, we have the cardinalities $(m_1, m_2, m_3, m_4) = (6, 3, 4, 1)$, and the edge matrix

$$\mathcal{E}_\pi = \begin{pmatrix} w & 6 & 12 & 6 \\ & 3 & 6 & 3 \\ & & 6 & 1 \\ & & & 0 \end{pmatrix}.$$

Now by Lemmas 4.4 and 4.5, we obtain $\det M(\pi, p, q) = -\frac{13718}{50625} w \geq 0$, so $w = 0$. Therefore, $\det M(\pi, p, q) = 0$, and computing the kernel of $M(\pi, p, q)$, we can set $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 4, 4, 4)$ in notations of Lemma 4.8, and apply that lemma for any $z \in G_{10}$ with $(e_2, e_3, e_4) = (0, 2, 0)$. We obtain $e_1 = 6$, that is the number of neighbors of z in G_{12} is exactly 6. So, G_{12} has no edges, and each vertex of $G_{10} \cup \{y\}$ is connected to all six vertices of G_{12}. Using the same argument as in the end of the subcase $(3, 1, 1, 1)$, we obtain that there are no edges in $G_{10} \cup \{y\}$, so $G_{12} \cup G_{10} \cup \{y\}$ is the required $K_{6,10}$ subgraph.

Now suppose that $y \in G_2$. Then the set G_{13} of neighbors of y in G_0 consists of exactly two vertices. Recalling that we are considering the subcase $(2, 2, 1, 1)$, there are three situations depending on the number of neighbors of G_{13} in G_3. Analogously to the notation for subcases, we denote these situations as $(2, 2)$, $(2, 1)$, and $(1, 1)$.

In the situation $(2, 2)$, we note that each of the three vertices of $G_{13} \cup \{y\}$ has exactly two neighbors in G_3 (recall that y has exactly 4 neighbors in $G_0 \cup G_3$, two of which are in G_0), so repeating the proof of subcase $(2, 2, 2, 0)$ with $H = G_{13} \cup \{y\} \cup G_0$, we can establish the existence of a 16-coclique in G.

In the situation $(2,1)$, (we temporarily use H, (b_j), (d_j) to denote different values from what they were assigned at the beginning of the current subcase) take $H = G_0 \cup G_3 \cup \{y\}$. Then $(d_j)_{j\geq 0} = (0,0,0,0,3,4,1,\dots)$. For any $x \in G \setminus H$, take $\pi = \{H, \{x\}\}$, let w be the number of neighbors of x in H. Then $\mathcal{E}_\pi = \left(\begin{smallmatrix} 19 & w \\ & 0 \end{smallmatrix}\right)$, and by Lemma 4.4 and Lemma 4.5, $\det M(\pi,p,q) = -\frac{1}{2025}(19w-56)^2 + \frac{2}{3} \geq 0$, so $2 \leq w \leq 4$, i.e., $b_0 = b_1 = b_5 = b_6 = \cdots = 0$. By Corollary 4.2 (v), we get $(b_j)_{j\geq 0} = (0,0,7,56,5,\dots)$. Let G_{14} be the set of 5 vertices of $G \setminus H$ each connected to exactly 4 vertices of H. Take $\pi = \{G_{14}, H\}$, then $\mathcal{E}_\pi = \left(\begin{smallmatrix} w & 20 \\ & 19 \end{smallmatrix}\right)$, where w is the number of edges in G_{14}. By Lemma 4.4 and Lemma 4.5, $\det M(\pi,p,q) = -\frac{76}{135}w + \frac{38}{81} \geq 0$, so $w = 0$. Let y_1 be the vertex of G_{13} with exactly two neighbors in G_3, we have 6 neighbors of y_1 in $H \setminus \{y_1\}$. Take $\pi = \{\{y_1\}, G_{14}, H \setminus \{y_1\}\}$, let w be the number of edges between y_1 and G_{14}, then

$$\mathcal{E}_\pi = \begin{pmatrix} 0 & w & 6 \\ & 0 & 20-w \\ & & 13 \end{pmatrix},$$

and by Lemma 4.4 and Lemma 4.5, $\det M(\pi,p,q) = -\frac{722}{6075}w^2 - \frac{1444}{3645}w \geq 0$, so $w = 0$. We have that $\{y_1\} \cup G_{14}$ is a 6-coclique, next we wish to find 10 vertices each connected to all vertices of $\{y_1\} \cup G_{14}$. Recalling that $b_3 = 7$, we denote by G_{15} the 7 vertices of $G \setminus H$ each having exactly 3 neighbors in H. Returning to our partition π, with $w = 0$ we have $\det M(\pi,p,q) = 0$, and computing the kernel of $M(\pi,p,q)$, we can set $(\lambda_1,\lambda_2,\lambda_3) = (5,1,4)$ in notations of Lemma 4.8, and apply that lemma for any $z \in G_{15}$ with either $(e_1,e_3) = (0,2)$ or $(e_1,e_3) = (1,1)$. We find that in the first case $e_2 = 6$, which is impossible, so must be in the second case, then $e_2 = 5$. Therefore, any $z \in G_{15}$ is connected to all vertices of $\{y_1\} \cup G_{14}$. It remains to find 3 more vertices to form the desired 10. The graph H has 3 vertices of degree four, denote them by G_{16}. We refine the partition π splitting $H \setminus \{y_1\}$ by redefining $\pi = \{\{y_1\}, G_{14}, G_{16}, H \setminus (\{y_1\} \cup G_{16})\}$ and apply Lemma 4.8 (with $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (5,1,4,4)$) for any $z \in G_{16}$ with either $(e_1,e_4) = (0,4)$ or $(e_1,e_4) = (1,3)$. We again obtain the impossible $e_2 = 6$ in the first case, so we must have $e_1 = 1$ and $e_2 = 5$, which shows that z is connected to all $\{y_1\} \cup G_{14}$. Using the same argument as in the end of the subcase $(3,1,1,1)$, we obtain that there are no edges in the subgraph $G_{15} \cup G_{16}$, so $\{y_1\} \cup G_{14} \cup G_{15} \cup G_{16}$ is the required $K_{6,10}$ subgraph.

In the situation $(1,1)$, we also take $H = G_0 \cup G_3 \cup \{y\}$, but now $(d_j)_{j\geq 0} = (0,0,0,0,2,6,\dots)$. As total number of edges in H is the same as in the situation $(2,1)$ (namely, 19), we argue similarly to obtain that $b_0 = b_1 = b_5 = b_6 = \cdots = 0$. By Corollary 4.2 (vi), we get $(b_j)_{j\ge 0} = (0, 0, 8, 54, 6, \dots)$. Let G_{17} be the 6 vertices of $G \setminus H$ each having exactly 4 neighbors in H, and let G_{18} be the 8 vertices of $G \setminus H$ each having exactly 2 neighbors in H. Take $\pi = \{G_{17}, H\}$, then $\mathcal{E}_\pi = \left(\begin{smallmatrix} w & 24 \\ & 19 \end{smallmatrix}\right)$, where w is the number of edges in G_{17}. By Lemma 4.4 and Lemma 4.5, $\det M(\pi, p, q) = -\frac{76}{135} w \ge 0$, so $w = 0$. Then we have $\det M(\pi, p, q) = 0$, and computing the kernel of $M(\pi, p, q)$, we can set $(\lambda_1, \lambda_2) = (1, 4)$ in notations of Lemma 4.8, and apply that lemma for any $z \in G_{18}$ with $e_2 = 2$ to obtain $e_1 = 6$. So, every vertex of G_{18} is connected to all vertices of G_{17}, which is a 6-coclique. We need two more such vertices, let G_{19} be the two vertices of H having degree 4 in H (one is y, and another one is in G_3 not connected to y). Take $\pi = \{G_{17}, H \setminus G_{19}, G_{19}\}$ and $(\lambda_1, \lambda_2, \lambda_3) = (1, 4, 4)$, apply Lemma 4.8 for any $z \in G_{19}$ with $e_2 = 4$ to get $e_1 = 6$. Therefore, any vertex of $G_{18} \cup G_{19}$ is connected to all vertices of G_{17}, hence, as in the end of proof of the subcase $(3, 1, 1, 1)$, the subgraph $G_{18} \cup G_{19}$ is a 10-coclique. In the summary, $G_{17} \cup G_{18} \cup G_{19}$ is the required $K_{6,10}$ subgraph.

This completes the treatment of the (subsub-) case $(b_j)_{j\ge 0} = (0, 0, 28, 40, 1, \dots)$ in the subcase $(2, 2, 1, 1)$, where, we recall that H (and, consequently, the corresponding (b_j) and (d_j)) was set as $H = G_0 \cup G_3$. So now we assume that $(b_j)_{j\ge 0} = (0, 1, 25, 43, 0, \dots)$ with this H. Define $y \in G \setminus H$ as the vertex with exactly one neighbor in H. Recall that $G_3 \subset G_2$ and G_2 is 10-regular. Hence, let G_{20} be the set of 24 vertices of $G_2 \setminus G_3$ that have exactly one edge to G_3 (each of the three vertices of G_3 is connected to some 8 vertices of $G_2 \setminus G_3$), note that there is no vertex of $G_2 \setminus G_3$ connected to more than one vertex of G_3 due to $b_4 = b_5 = 0$. Clearly $y \in G_1$, and by (5.1), there are 18 edges from y to G_2, and in particular, at least 6 vertices of G_{20} are not connected to y. Let G_{21} be any such 6 vertices. Take $\pi = \{G_{21}, G_3, G_0, \{y\}\}$, let w be the number of edges in G_{21}, then

$$\mathcal{E}_\pi = \begin{pmatrix} w & 6 & 12 & 0 \\ & 3 & 6 & 0 \\ & & 6 & 1 \\ & & & 0 \end{pmatrix},$$

and by Lemma 4.4 and Lemma 4.5, we obtain $\det M(\pi, p, q) = -\frac{13718}{50625} w \ge 0$, so $w = 0$. Therefore, $\det M(\pi, p, q) = 0$, and computing the kernel of $M(\pi, p, q)$, we can set $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 4, 6, -4)$ in notations of Lemma 4.8. Note that among 43 vertices with exactly 3 neighbors in H, there are 24 (G_{20}) from G_2, and hence 19 from G_1. But as G_1 is 11-regular, there are at least

$19-11=8$ vertices from these 19 not connected to y. Denote any set of such 8 vertices as G_{22}. Now we apply Lemma 4.8 for any $z \in G_{22}$ with $(e_2,e_3,e_4)=(2,1,0)$ and get $e_1=0$, so there are no edges from G_{22} to G_{21}. Let G_{23} be the subgraph consisting of one vertex that has degree 5 in H and is not connected to y. Refining π as $\pi=\{G_{21},G_3,G_0\setminus G_{23},G_{23},\{y\}\}$ and taking $(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)=(1,4,6,6,-4)$ we apply Lemma 4.8 for $z\in G_{23}$ with $(e_2,e_3,e_5)=(2,3,0)$ to get $e_1=0$, so, G_{23} is not connected to G_{21}. Next we take $\pi=\{G_{22},G_3,G_0,\{y\}\}$, denote by w the number of edges in G_{22}, observe that

$$\mathcal{E}_\pi = \begin{pmatrix} w & 16 & 8 & 0 \\ & 3 & 6 & 0 \\ & & 6 & 1 \\ & & & 0 \end{pmatrix},$$

and by Lemma 4.4 and Lemma 4.5, we obtain $\det M(\pi,p,q) = -\frac{13718}{50625}w+\frac{109744}{455625}\geq 0$, so $w=0$. To show that $\{y\}\cup G_{21}\cup G_{22}\cup G_{23}$ is a 16-coclique, it only remains to verify that there are no edges between G_{22} and G_{23}. This is quite straightforward using already established structure. Indeed, take $\pi=\{G_{23},G_{22},G_3,G_0\setminus G_{23},\{y\}\}$, set w to be the number of edges between G_{23} and G_{22}, then

$$\mathcal{E}_\pi = \begin{pmatrix} 0 & w & 2 & 3 & 0 \\ & 0 & 16 & 8-w & 0 \\ & & 3 & 4 & 0 \\ & & & 3 & 1 \\ & & & & 0 \end{pmatrix},$$

and by Lemma 4.4 and Lemma 4.5, we obtain $\det M(\pi,p,q) = -\frac{130321}{2278125}w^2-\frac{4170272}{20503125}w\geq 0$, so $w=0$. This completes the proof of the subcase $(2,2,1,1)$, and thus, of the lemma. ☐

6. The case of $SRG(40,12,2,4)$

Proof of Lemma 5.2. Suppose G is a $SRG(76,30,8,14)$, $\widetilde{G}\subset G$ is a $SRG(40,12,2,4)$, for any $z\in G\setminus\widetilde{G}$ both $N(z)\cap\widetilde{G}$ and $N'(z)\cap\widetilde{G}$ are 4-regular subgraphs on 20 vertices, and that $|N(z_1)\cap N(z_2)\cap\widetilde{G}|=8$ for any adjacent $z_1,z_2\in G\setminus\widetilde{G}$. By Lemma 4.15, $\operatorname{rank}B(\widetilde{G})=\operatorname{rank}(\operatorname{lin}(\{x_i,\, i\in\widetilde{G}\}))=16$, where $x_i\in\mathbb{R}^{18}$ is the Euclidean representation of $i\in G$. For $j\in G\setminus\widetilde{G}$, denote by x_j' the projection of x_j onto $\operatorname{lin}\{x_i, i\in\widetilde{G}\}$. For $j^{(1)},j^{(2)}\in G\setminus\widetilde{G}$, our goal is to find the dot products $x''_{j^{(1)}}\cdot x''_{j^{(2)}}$, where $x''_j=x_j-x'_j$ is the projection of x_j onto the orthogonal complement of $\operatorname{lin}\{x_i,i\in\widetilde{G}\}$, which is a $18-16=2$-dimensional Euclidean space.

Fix $j \in G \setminus \widetilde{G}$. Apply Proposition 4.11 to the equitable partition $\pi = \{N(j) \cap \widetilde{G}, N'(j) \cap \widetilde{G}\}$, where by assumption $|N(j) \cap \widetilde{G}| = |N'(j) \cap \widetilde{G}| = 20$ and the degree matrix is $\mathcal{D}_\pi = \left(\begin{smallmatrix} 4 & 8 \\ 8 & 4 \end{smallmatrix}\right)$. Solving the system (4.7) with $(a_1, a_2) = (1, 0)$, we get $(\alpha_1, \alpha_2) = (-\frac{1}{9}, \frac{1}{18})$, which means

$$x'_j = -\frac{1}{9} \sum_{i \in N(j) \cap \widetilde{G}} x_i + \frac{1}{18} \sum_{i \in N'(j) \cap \widetilde{G}} x_i.$$

Now take $j^{(1)}, j^{(2)} \in G \setminus \widetilde{G}$, denote the corresponding partitions by $\pi^{(1)}$ and $\pi^{(2)}$, and apply Proposition 4.12. To simplify the right-hand side of (4.8), we first note that by strong regularity of $\widetilde{G}$ and our assumption all values $|G_{u,w}|$, $e_{u,w}$ and $e_{u,w,\tilde{u},\tilde{w}}$ can be computed in terms of $n_{j^{(1)},j^{(2)}} := |G_{1,1}| = |G_1^{(1)}, G_1^{(2)}| = |N(j^{(1)}) \cap N(j^{(2)}) \cap \widetilde{G}|$ and $e_{j^{(1)},j^{(2)}} := e_{1,1}$, the number of edges in $G_{1,1}$. Indeed, it straightforward that $|G_{1,1}| = |G_{2,2}| = n_{j^{(1)},j^{(2)}}$, $|G_{1,2}| = |G_{2,1}| = 20 - n_{j^{(1)},j^{(2)}}$, $e_{1,1} = e_{j^{(1)},j^{(2)}}$, $e_{1,2} = e_{2,1} = 40 - e_{j^{(1)},j^{(2)}}$, $e_{2,2} = 8n_{j^{(1)},j^{(2)}} - e_{j^{(1)},j^{(2)}}$, $e_{1,1,1,2} = e_{1,1,2,1} = e_{1,2,1,1} = e_{2,1,1,1} = 4n_{j^{(1)},j^{(2)}} - 2e_{j^{(1)},j^{(2)}}$, $e_{1,1,2,2} = e_{2,2,1,1} = 4n_{j^{(1)},j^{(2)}} + 2e_{j^{(1)},j^{(2)}}$, $e_{1,2,2,2} = e_{2,1,2,2} = e_{2,2,1,2} = e_{2,2,2,1} = -4n_{j^{(1)},j^{(2)}} + 2e_{j^{(1)},j^{(2)}}$, and finally $e_{1,2,2,1} = e_{2,1,1,2} = 160 - 12n_{j^{(1)},j^{(2)}} + 2e_{j^{(1)},j^{(2)}}$. Non-negativity of the above values implies that $e_{j^{(1)},j^{(2)}} = 2n_{j^{(1)},j^{(2)}}$, but even without use of this relation, simplifying the right hand side of (4.12) we obtain

$$x'_{j^{(1)}} \cdot x'_{j^{(2)}} = \frac{19}{270} n_{j^{(1)},j^{(2)}} - \frac{52}{81}.$$

Our construction yields $n_{j^{(1)},j^{(2)}} = 20$ if $j^{(1)} = j^{(2)}$, so with the above notations we can apply Proposition 4.13 to see that all projections x''_j, $j \in G \setminus \widetilde{G}$, have the same Euclidean norm, which means they belong to a (2-dimensional, planar) circle. For convenience, we define the normalized projections $x'''_j := \frac{x''_j}{\|x''_j\|}$.

Next, using (4.10), if $j^{(1)}$ and $j^{(2)}$ are adjacent, then (by assumption) $n_{j^{(1)},j^{(2)}} = 8$, so $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} = -\frac{4}{5}$. If $j^{(1)}$ and $j^{(2)}$ are disjoint, then $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} = -\frac{3}{10} n_{j^{(1)},j^{(2)}} + \frac{17}{5}$. Fix $j \in G \setminus \widetilde{G}$. Take any $j^{(1)}, j^{(2)} \in G \setminus \widetilde{G}$ such that $x'''_j \cdot x'''_{j^{(1)}} = -\frac{4}{5}$ and $x'''_j \cdot x'''_{j^{(2)}} = -\frac{4}{5}$ (for example, j can be adjacent to both $j^{(1)}$ and $j^{(2)}$). We can assume that x'''_j is $(1, 0)$ in $\mathbb{R}^2$, then each of $x'''_{j^{(1)}}$ and $x'''_{j^{(2)}}$ can be either $(-\frac{4}{5}, \frac{3}{5})$ or $(-\frac{4}{5}, -\frac{3}{5})$. We claim that both possibilities cannot be attained. Indeed, otherwise $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} = \frac{7}{25}$ leading to $n_{j^{(1)},j^{(2)}} = \frac{52}{5}$, which is not integer. So, all $i \in G \setminus \widetilde{G}$ such that $x'''_i \cdot x'''_j = -\frac{4}{5}$ (in particular, neighbors i of j) have the same projection x'''_i, which is either $(-\frac{4}{5}, \frac{3}{5})$ or $(-\frac{4}{5}, -\frac{3}{5})$. Now, if $i \in G \setminus \widetilde{G}$ is disjoint with j, but they both have a common neighbor in $G \setminus \widetilde{G}$, we use the above argument for that neighbor to get that $x'''_i = x'''_j$. But if $i \in G \setminus \widetilde{G}$ is disjoint with j and has no common neighbors in $G \setminus \widetilde{G}$, then all common neighbors are in $\widetilde{G}$, hence $n_{i,j} = 14$, and so $x_i''' \cdot x_j''' = -\frac{4}{5}$. In summary, $\{x_i''', i \in G \setminus \widetilde{G}\}$ attains only two values: $(1, 0)$ and one of $(-\frac{4}{5}, \pm\frac{3}{5})$.

But then clearly

$$\sum_{i \in G \setminus \widetilde{G}} x_i''' \neq (0, 0).$$

On the other hand, $\sum_{i \in G} x_i = 0$ and $x_i'' = (0, 0)$ for $i \in \widetilde{G}$, so

$$\sum_{i \in G \setminus \widetilde{G}} x_i'' = (0, 0),$$

which is a contradiction that completes the proof of the lemma. □

7. The case of 16-coclique

Proof of Lemma 5.3. Suppose $\widetilde{G}$ is a 16-coclique in G, which is $SRG(76, 30, 8, 14)$. Similarly to the proof of Lemma 5.2, we apply Lemma 4.14 to see that $\text{rank}B(\widetilde{G}) = \text{rank}(\text{lin}(\{x_i, i \in \widetilde{G}\})) = 16$, where $x_i \in \mathbb{R}^{18}$ is the Euclidean representation of $i \in G$. For $j \in G \setminus \widetilde{G}$, denote by x_j' the projection of x_j onto $\text{lin}\{x_i, i \in \widetilde{G}\}$. For $j^{(1)}, j^{(2)} \in G \setminus \widetilde{G}$, we will compute the dot products $x''_{j^{(1)}} \cdot x''_{j^{(2)}}$, where $x_j'' = x_j - x_j'$ is the projection of x_j onto the orthogonal complement of $\text{lin}\{x_i, i \in \widetilde{G}\}$, which is a $18 - 16 = 2$-dimensional Euclidean space.

Next we claim that for any $j \in G \setminus \widetilde{G}$, we have $|N(j) \cap \widetilde{G}| = |N'(j) \cap \widetilde{G}| = 8$. It is convenient to use the dual Euclidean representation of G, namely for any $i \in G$ there exists $z_i \in \mathbb{R}^{57}$ satisfying (2.8) with $(p, q) = (\frac{1}{15}, -\frac{1}{15})$. Then $\left(\sum_{i \in \widetilde{G}} z_i\right)^2 = 16 + 16 \cdot 15 \cdot \frac{-1}{15} = 0$, hence $0 = z_j \cdot \left(\sum_{i \in \widetilde{G}} z_i\right) = \frac{1}{15}(|N(j) \cap \widetilde{G}| - |N'(j) \cap \widetilde{G}|)$, and the claim follows.

For a fixed $j \in G \setminus \widetilde{G}$ we apply Proposition 4.11 to the equitable partition $\pi = \{N(j) \cap \widetilde{G}, N'(j) \cap \widetilde{G}\}$, where $|N(j) \cap \widetilde{G}| = |N'(j) \cap \widetilde{G}| = 8$ and the degree matrix is trivially $\mathcal{D}_\pi = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Solving the system (4.7) with $(a_1, a_2) = (1, 0)$, we get $(\alpha_1, \alpha_2) = (-\frac{4}{15}, \frac{7}{30})$. Now take $j^{(1)}, j^{(2)} \in G \setminus \widetilde{G}$, denote the corresponding partitions by $\pi^{(1)}$ and $\pi^{(2)}$, and apply Proposition 4.12. We clearly have that all $e_{u,w}$ and $e_{u,w,\tilde{u},\tilde{w}}$ are zeroes, and with $n_{j^{(1)},j^{(2)}} := |G_{1,1}|$ we immediately obtain $|G_{1,1}| = |G_{2,2}| = n_{j^{(1)},j^{(2)}}$, $|G_{1,2}| = |G_{2,1}| = 8 - n_{j^{(1)},j^{(2)}}$. Now by (4.8),

$$(x'_{j^{(1)}}, x'_{j^{(2)}}) = \frac{19}{90} n_{j^{(1)},j^{(2)}} - \frac{112}{135}.$$

If $j^{(1)} = j^{(2)}$, then $n_{j^{(1)},j^{(2)}} = 8$, so with the above notations we can apply Proposition 4.13 to see that all projections x_j'', $j \in G \setminus \widetilde{G}$, have the same Euclidean norm, which means they belong to a (2-dimensional, planar) circle. For convenience, we define the normalized projections $x_j''' := \frac{x_j''}{\|x_j''\|}$.

Using (4.10), if $a \in \{0,1\}$ is the number of edges (adjacency) between $j^{(1)}$ and $j^{(2)}$, then $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} = -\frac{3}{2}n_{j^{(1)},j^{(2)}} + 7 - a \in [-1,1]$, which, as $n_{j^{(1)},j^{(2)}}$ is integer, leads to one of the following four possibilities:

$$x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} = \begin{cases} 1, & \text{if } n_{j^{(1)},j^{(2)}} = 2 \text{ and } a = 1, \\ -\frac{1}{2}, & \text{if } n_{j^{(1)},j^{(2)}} = 3 \text{ and } a = 1, \\ 1, & \text{if } n_{j^{(1)},j^{(2)}} = 4 \text{ and } a = 0, \\ -\frac{1}{2}, & \text{if } n_{j^{(1)},j^{(2)}} = 5 \text{ and } a = 0. \end{cases} \tag{7.1}$$

In particular, $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} \in \{1, -\frac{1}{2}\}$, so that there are only three possible values for x_j''', $j \in G \setminus \widetilde{G}$, which are the vertices of an equilateral triangle inscribed into the unit circle. Now let $\{H_1, H_2, H_3\}$ be the partition of $G \setminus \widetilde{G}$ such that the value of x_j''' is the same for any j in one component of the partition. Without loss of generality, $x_j''' = (\cos(t\pi/3), \sin(t\pi/3))$, $j \in H_t$, $t = 1, 2, 3$. Arguing as in the end of the proof of Lemma 5.2, we have $\sum_{j \in G\setminus\widetilde{G}} x_j''' = (0,0)$, which implies $|H_1| = |H_2| = |H_3| = 20$.

It is sufficient to work with H_1, but the same statements are valid for the other two components of the partition.

First we show that H_1 is 2-regular. For any $i \in \widetilde{G}$, we have $N(i) \subset G\setminus\widetilde{G}$ and $|N(i)| = 30$. We claim that $|N(i) \cap H_1| = 10$. By applying projections to (2.10), we have that $\sum_{j\in N(i)} x_j''' = (0,0)$, therefore there will be equal number of elements of $N(i)$ in each part H_1, H_2, and H_3, in particular, $|N(i) \cap H_1| = 10$. Next let w be the number of edges in H_1, then computing the number of paths of length 2 originating and terminating in H_1 going through $\widetilde{G}$, we have $16 \cdot \frac{10\cdot 9}{2} = w \cdot 2 + (190 - w) \cdot 4$, so $w = 20$. Using the Euclidean representation of G in $\mathbb{R}^{57}$, $(\sum_{i\in H_1} z_i)^2 = 20 + 40 \cdot \frac{1}{15} + 340 \cdot \frac{-1}{15} = 0$, hence for any $j \in H_1$ we have $z_j \cdot (\sum_{i\in H_1} z_i) = 1 + \frac{1}{15}|N(j) \cap H_1| + \frac{-1}{15}(19 - |N(j) \cap H_1|)$, implying that $|N(j) \cap H_1| = 2$, so H_1 is 2-regular. Any 2-regular graph is a union of cycles.

Next we show that if C_l is a cycle of length l in H_1, then for any $i \in \widetilde{G}$, we have $|N(i) \cap C_l| = l/2$, in particular l is even and is not less than 4 (there is no cycle of length 2). We know that $|N(i) \cap H_1| = 10$, so if H_1 consists only of one cycle, we are done. Otherwise, it is enough to show for any two cycles C_{l_1} and C_{l_2} in H_1 of lengths l_1 and l_2 respectively, we have

$l_1/l_2 = |N(i) \cap C_{l_1}|/|N(i) \cap C_{l_2}|$, i.e., the lengths are proportional to the number of neighbors (the sum of the lengths is 20 and the total number of neighbors is 10). Let $a_t = |N(i) \cap C_{l_t}|$, $t = 1, 2$. We use the Euclidean representation of G in $\mathbb{R}^{18}$, recall that then $(p, q) = (-\frac{4}{15}, \frac{7}{45})$, hence, $1 + 2p - 3q = 0$. Therefore,

$$\left(l_2 \sum_{j \in C_{l_1}} x_j - l_1 \sum_{j \in C_{l_2}} x_j \right)^2 = l_2^2(l_1 + 2l_1 p + (l_1^2 - 3l_1)q) - 2l_1^2 l_2^2 q + l_1^2(l_2 + 2l_2 p + (l_2^2 - 3l_2)q) = 0,$$

and

$$0 = x_i \cdot \left(l_2 \sum_{j \in C_{l_1}} x_j - l_1 \sum_{j \in C_{l_2}} x_j \right) = l_2(a_1 p + (l_1 - a_1)q) - l_1(a_2 p + (l_2 - a_1)q) = (l_2 a_1 - l_1 a_2)p,$$

yielding the desired $l_1/l_2 = a_1/a_2$.

We are now in position to use Lemma 4.16. As all projections x_j'', $j \in H_1$, are the same, and they are projections onto a 2-dimensional subspace of $\mathbb{R}^{18}$, we have $\mathrm{rank}(\mathrm{lin}(\{x_j,\, j \in H_1\})) \leq 17$, so by Lemma 4.16, there are at least 4 cycles in H_1. Therefore, there are only the following three possibilities for the lengths of the cycles: 5 cycles of length 4, or two cycles of length 6 and two cycles of length 4, or one cycle of length 8 and three cycles of length 4. In either of the cases, there is a cycle $C_4 \subset H_1$ of length 4, which will suffice for us to complete the proof.

Suppose that $\widetilde{G} = \{g_1, \ldots, g_{16}\}$. For $i \in H_1$, define $A(i)$ as the 8-element subset of $\{1, 2, \ldots, 16\}$ such that $N(i) \cap \widetilde{G} = \{g_t : t \in A(i)\}$. By (7.1), if $i, j \in H_1$ are adjacent, then $|A(i) \cap A(j)| = 2$; and if $i, j \in H_1$ are disjoint, then $|A(i) \cap A(j)| = 4$. It is not hard to see that without loss of generality (by permutation of indexes) we can assume that our C_4 has the following representation:

$$\{A(i) : i \in C_4\} = \{\{1, 2, 3, 4, 5, 6, 7, 8\}, \{1, 2, 9, 10, 11, 12, 13, 14\},$$
$$\{5, 6, 7, 8, 13, 14, 15, 16\}, \{3, 4, 9, 10, 11, 12, 15, 16\}\}.$$

Now let $\mathfrak{M}$ be the collection of all 8-element subsets of $\{1, 2, \ldots, 16\}$, then $|\mathfrak{M}| = \binom{16}{8} = 12870$. Consider the following graph on $\mathfrak{M}$: two vertices $A_1, A_2 \in \mathfrak{M}$ are adjacent if and only if $|A_1 \cap A_2|$ is either 2 or 4. We fix $\mathfrak{M}_0 := \{A(i) : i \in C_4\}$, $|\mathfrak{M}_0| = 4$, and define $\mathfrak{M}_1 := \{A \in \mathfrak{M} : \mathfrak{M}_0 \subset N(A)\}$, where $N(A)$ denotes all neighbors in our graph on $\mathfrak{M}$. Clearly, $\{A(i) : i \in H_1 \setminus C_4\}$ is a 16-clique in $\mathfrak{M}_1$. We obtain a contradiction by showing that the largest clique in $\mathfrak{M}_1$ has size 15.

To this end, we use the mathematical software Sage, in particular, the function `clique_number` returning the order of the largest clique of the given graph, which is based on the Bron-Kerbosch algorithm [BK73]. Note that $\mathfrak{M}_1$ can be easily generated, it has 906 vertices and 176672 edges. The procedure's running time is well under one hour on a modern personal computer. See [BPR] for the source code and the output. $\square$

Remark 7.1. One can use the second cycle of length four to reduce the problem to graphs of smaller size that would not require the use of the more sophisticated algorithms for the computation of the largest clique. However, this would lead to a more complicated programming and longer running time.

8. The case of $K_{6,10}$

Proof of Lemma 5.4. Let $\widetilde{G}$ is a $K_{6,10}$ and a subgraph of G, which is $SRG(76, 30, 8, 14)$. Let $\widetilde{G}_1$ be the 6-coclique in $\widetilde{G}$, and $\widetilde{G}_2$ be the 10-coclique in $\widetilde{G}$, so that $\widetilde{G} = \widetilde{G}_1 \cup \widetilde{G}_2$. As before, we apply Lemma 4.14 to see that $\operatorname{rank} B(\widetilde{G}) = \operatorname{rank}(\operatorname{lin}(\{x_i,\, i \in \widetilde{G}\})) = 15$, where $x_i \in \mathbb{R}^{18}$ is the Euclidean representation of $i \in G$. For $j \in G \setminus \widetilde{G}$, denote by x'_j the projection of x_j onto $\operatorname{lin}\{x_i, i \in \widetilde{G}\}$. For $j^{(1)}, j^{(2)} \in G \setminus \widetilde{G}$, we will compute the dot products $x''_{j^{(1)}} \cdot x''_{j^{(2)}}$, where $x''_j = x_j - x'_j$ is the projection of x_j onto the orthogonal complement of $\operatorname{lin}\{x_i, i \in \widetilde{G}\}$, which is a $18 - 15 = 3$-dimensional Euclidean space.

Next we show that for any $j \in G \setminus \widetilde{G}$, we have $|N(j) \cap \widetilde{G}_1| = 2$ and $|N(j) \cap \widetilde{G}_2| = 4$. The partition $\pi = \{\widetilde{G}_1, \widetilde{G}_2$ of $\widetilde{G}$ has edge matrix $\mathcal{E}_\pi = \left(\begin{smallmatrix} 0 & 60 \\ 60 & 0 \end{smallmatrix}\right)$. Then, with $(p, q) = (-\frac{4}{15}, \frac{7}{45})$, in the notations of Lemma 4.4, $\det M(\pi, p, q) = 0$. Therefore, we can compute the kernel of $M(\pi, p, q)$, set $(\lambda_1, \lambda_2) = (1, 2/3)$ and $z = j$ in notations of Lemma 4.8. By (4.4) of that lemma, $57|N(j) \cap \widetilde{G}_1| + 38|N(j) \cap \widetilde{G}_1| = 266$. We apply the same procedure for the dual representation, with $(p, q) = (\frac{1}{15}, -\frac{1}{15})$, and obtain another linear equation $-|N(j) \cap \widetilde{G}_1| + |N(j) \cap \widetilde{G}_2| = 2$. The two equations immediately lead to the claimed $|N(j) \cap \widetilde{G}_1| = 2$ and $|N(j) \cap \widetilde{G}_2| = 4$. Note that during the proof we established that $\sum_{i \in \widetilde{G}_1} x_i + \frac{2}{3} \sum_{i \in \widetilde{G}_2} x_i = 0$.

Now, with fixed $j \in G \setminus \widetilde{G}$, we will apply Proposition 4.11 to the equitable partition $\pi = \{N(j) \cap \widetilde{G}_1, N'(j) \cap \widetilde{G}_1, N(j) \cap \widetilde{G}_2, N'(j) \cap \widetilde{G}_2\}$, where the degree matrix is

$$\mathcal{D}_\pi = \left(\begin{smallmatrix} 0 & 0 & 4 & 6 \\ 0 & 0 & 4 & 6 \\ 2 & 4 & 0 & 0 \\ 2 & 4 & 0 & 0 \end{smallmatrix}\right).$$

Due to linear dependence $\sum_{i \in \widetilde{G}_1} x_i + \frac{2}{3} \sum_{i \in \widetilde{G}_2} x_i = 0$, we can expect a one-parametric family of the solutions of the system (4.7) with $(a_1, a_2, a_3, a_4) = (1, 0, 1, 0)$. Indeed, we get

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\frac{3}{2}r - \frac{5}{8}, \frac{3}{2}r - \frac{1}{8}, r - \frac{1}{2}, r)$, $r \in \mathbb{R}$, and it is convenient to assume that $\alpha_1 = \alpha_3$, which is achieved by taking $r = \frac{1}{4}$, then $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4})$, so we also have that $\alpha_2 = \alpha_4$. We established that

$$x'_j = -\frac{1}{4} \sum_{i \in N(j) \cap \widetilde{G}} x_i + \frac{1}{4} \sum_{i \in N'(j) \cap \widetilde{G}} x_i. \tag{8.1}$$

Now take $j^{(1)}, j^{(2)} \in G \setminus \widetilde{G}$, define the partitions $\pi^{(1)} = \{N(j^{(1)}) \cap \widetilde{G}, N'(j^{(1)}) \cap \widetilde{G}\}$ and $\pi^{(2)} = \{N(j^{(2)}) \cap \widetilde{G}, N'(j^{(2)}) \cap \widetilde{G}\}$, and apply Proposition 4.12. To simplify the right-hand side of (4.8), we use two variables: $n_1 = |N(j^{(1)}) \cap N(j^{(2)}) \cap \widetilde{G}_1|$ and $n_2 = |N(j^{(1)}) \cap N(j^{(2)}) \cap \widetilde{G}_2|$. The structure of $\widetilde{G}$ then implies $n_{j^{(1)},j^{(2)}} := |G_{1,1}| = n_1 + n_2$, $G_{1,2} = G_{2,1} = 6 - n_1 - n_2$, $4 + n_1 + n_2$, $e_{1,1} = n_1 n_2$, $e_{1,2} = e_{2,1} = (2 - n_1)(4 - n_2)$, $e_{2,2} = (2 + n_1)(2 + n_2)$, $e_{1,1,1,2} = e_{1,1,2,1} = e_{1,2,1,1} = e_{2,1,1,1} = n_1(4 - n_2) + n_2(2 - n_1)$, $e_{1,1,2,2} = e_{2,2,1,1} = n_1(2 + n_2) + n_2(2 + n_1)$, $e_{1,2,2,2} = e_{2,1,2,2} = e_{2,2,1,2} = e_{2,2,2,1} = (2 - n_1)(2 + n_2) + (4 - n_2)(2 + n_1)$, and finally $e_{1,2,2,1} = e_{2,1,1,2} = 2(2 - n_1)(4 - n_2)$. Simplifying the right hand side of (4.12), we obtain

$$x'_{j^{(1)}} \cdot x'_{j^{(2)}} = \frac{19}{90} n_{j^{(1)},j^{(2)}} - \frac{43}{90}.$$

We have $n_{j^{(1)},j^{(2)}} = 6$ if $j^{(1)} = j^{(2)}$, so with the above notations we can apply Proposition 4.13 to see that all projections x''_j, $j \in G \setminus \widetilde{G}$, have the same Euclidean norm, which means they belong to an Euclidean sphere in three dimensions. More specifically, by (8.1) we have $(x_j, x'_j) = -\frac{1}{4} \cdot 6p + \frac{1}{4} \cdot 10q = \frac{71}{90}$, therefore $\|x''_j\|^2 = \|x_j - x'_j\|^2 = 1 - 2\frac{71}{90} + \frac{19}{90}6 - \frac{43}{90} = \frac{19}{90}$. For convenience, we define the normalized projections $x'''_j := \frac{x''_j}{\|x''_j\|}$.

Next, using (4.10), if $j^{(1)}$ and $j^{(2)}$ are disjoint, then $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} = -n_{j^{(1)},j^{(2)}} + 3$. If $j^{(1)}$ and $j^{(2)}$ are adjacent, then $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} = -n_{j^{(1)},j^{(2)}} + 1$. Since $n_{j^{(1)},j^{(2)}}$ is an integer, this implies that $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}}$ can only take one of the three values from $\{-1, 0, 1\}$.

Therefore, it is easy to see that the possible values of x'''_j, $j \in G \setminus \widetilde{G}$, are vertices of an octahedron in $\mathbb{R}^3$, so without loss of generality we can assume that $x'''_j \in \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$, $j \in G \setminus \widetilde{G}$. Let $H_1 = \{i \in G \setminus \widetilde{G} : x'''_i = (1, 0, 0)\}$ and $H_2 = \{i \in G \setminus \widetilde{G} : x'''_i = (-1, 0, 0)\}$. Arguing as in the end of the proof of Lemma 5.2, we have $\sum_{j \in G \setminus \widetilde{G}} x'''_j = (0, 0, 0)$. Then clearly $|H_1| = |H_2|$. We use (2.9) with $y = x'''_i$ for some fixed $i \in H_1$. Clearly, $x_j \cdot x'''_i = 0$ for $j \in \widetilde{G}$.

On the other hand, for $j \in \widetilde{G}$, we have (recall that $\|x_i''\| = \|x_j''\| = \sqrt{\frac{19}{90}}$)

$$x_j \cdot x_i''' = \frac{1}{\|x_i''\|}(x_j' + x_j'') \cdot x_i'' = \frac{x_j'' \cdot x_i''}{\|x_i''\|} = \|x_i''\| x_j''' \cdot x_i''' = \begin{cases} \sqrt{\frac{19}{90}}, & \text{if } j \in H_1, \\ -\sqrt{\frac{19}{90}}, & \text{if } j \in H_2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, by (2.9), $\frac{19}{90}(|H_1| + |H_2|) = \frac{76}{18}$, so $|H_1| = |H_2| = 10$.

We keep $i \in H_1$ fixed, and choose arbitrary $t \in H_1$. By (2.10) for x_t, we have $x_t + \frac{1}{8}\sum_{j \in N(t)} x_j = 0$, and multiplying by x_i''', we obtain $\sqrt{\frac{19}{90}}(1 + \frac{1}{8}(|N(t) \cap H_1| - |N(t) \cap H_2|)) = 0$, so $|N(t) \cap H_1| = 8 + |N(t) \cap H_2|$ for any $t \in H_1$. Similarly, $|N(t) \cap H_2| = 8 + |N(t) \cap H_1|$ for any $t \in H_2$.

There is only a finite number of possible subgraphs $H_1 \cup H_2$ satisfying $|H_1| = |H_2| = 10$ and the conditions that $|N(t) \cap H_1| = 8 + |N(t) \cap H_2|$ for any $t \in H_1$ and $|N(t) \cap H_2| = 8 + |N(t) \cap H_1|$ for any $t \in H_2$. The main idea for the completion of the proof is to verify that all (or almost all) such subgraphs $U = H_1 \cup H_2$ would fail to satisfy the conditions of Proposition 2.1, which is done with an assistance of a computer algebra system.

To generate all such possible subgraphs, we observe that if we invert the edges between H_1 and H_2, we obtain a regular graph of degree 2. Indeed, for each vertex t in H_1 there can be either 0, 1, or 2 edges to other vertices in H_1 (because $|N(t) \cap H_1| = 8 + |N(t) \cap H_2| \le |H_2| = 10$). Then the number of edges from t to H_2 in the inverted graph is $10 - |N(t) \cap H_2| = 2 - |N(t) \cap H_1|$, which is 2, 1, or 0, respectively. Any regular graph of degree 2 is a union of cycles, which significantly simplifies generation of all required subgraphs.

If w is the number of edges in H_1, then there are $80 + 2w$ edges between H_1 and H_2, and, hence also w edges in H_2. With $\pi := \{H_1, H_2\}$, the edge matrix is $\mathcal{E}_\pi = \left(\begin{smallmatrix} w & 80+2w \\ & w \end{smallmatrix}\right)$, and by Lemma 4.5, $-\frac{5776}{81}w + \frac{19760}{81} \ge 0$, so $w \le 3$.

Now let us briefly describe the computer verification that there can be no subgraphs H_1 and H_2 satisfying the above restrictions. We begin with generating all possible decompositions of 20 into the sum of integers not smaller than 3. This gives all possibilities for decomposition of $H_1 \cup H_2$ with inverted edges into the union of cycles. For each possible length of cycle (between 3 and 20), we generate all possibilities of assigning a vertex to either H_1 or H_2, ignoring all assignments where the total number of consecutive pairs of vertices assigned to the same subgraph H_1 or H_2 exceeds 3 (each such pair gives an edge). We need to generate only non-isomorphic assignments, which reduces the number of required possibilities. Next we combine the prepared data, generate $H_1 \cup H_2$ as union of cycles, invert edges between H_1 and H_2, and for each resulting possibility we perform three checks: (i) the number of edges in H_1 is equal to the number of edges in H_2 and does not exceed 3; (ii) the rank of $(x_i \cdot x_j)_{i,j \in H_1 \cup H_2}$ does not exceed 16; (iii) the smallest eigenvalue of $(x_i \cdot x_j)_{i,j \in H_1 \cup H_2}$ is non-negative. The conditions (ii) and (iii) must be valid by Proposition 2.1. There will be only four cases when all of the above conditions are satisfied, namely, when there are five cycles of length 4.

To handle the remaining cases, we show that there is a vertex $t \in \widetilde{G}_1$ such that $|N(t) \cap H_1| = |N(t) \cap H_2| \leq 3$. Then, as verified by the computer, it turns out that the rank of $(x_i \cdot x_j)_{i,j \in H_1 \cup H_2 \cup \{t\}}$ is at least 17, which is a contradiction. We want to remark that the computations needed for this lemma take less than 15 minutes on a modern personal computer. It only remains to justify existence of $t \in \widetilde{G}_1$ such that $|N(t) \cap H_1| = |N(t) \cap H_2| \leq 3$. First, let $t \in \widetilde{G}_1$ be arbitrary. By (2.10) for x_t, we have $x_t + \frac{1}{8} \sum_{j \in N(t)} x_j = 0$, and multiplying by x_i''', where $i \in H_1$, we obtain $|N(t) \cap H_1| = |N(t) \cap H_2|$. But recall that for any vertex $j \in H_1 \cup H_2$, we have $|N(j) \cap \widetilde{G}_1| = 2$, so there are 40 edges between $H_1 \cup H_2$ and $\widetilde{G}_1$. Hence, there must be $t \in \widetilde{G}_1$ with no more than $\frac{40}{6}$ neighbors in $H_1 \cup H_2$, and the claim follows. □

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